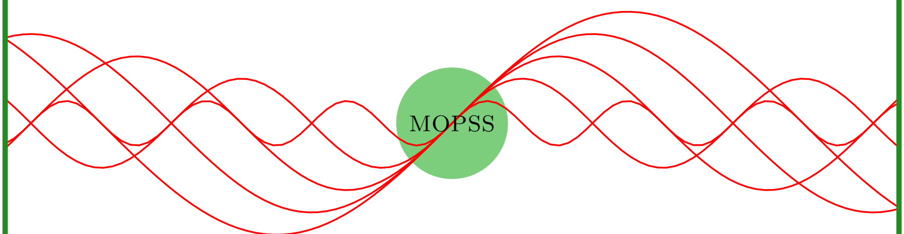
Pigeonhole principle

MOPSS

21 February 2025

Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Pigeonhole principle

See [**Sob13**, Chapter 2], [**Mat24**, Chapter 6].

Example 1.1. Let $X = \{a_1, a_2, \ldots, a_5\}$ where $a_1, a_2, \ldots, a_5$ are integers which are perfect squares. Show that there exists a subset $Y = \{b_1, b_2, b_3\}$ of X such that $b_1 + b_2 + b_3$ is divisible by 3.

Solution 1. Since perfect squares are congruent to 0 or 1 modulo 3, at least three elements b_1, b_2, b_3 of X are congruent to either 0 modulo 3, or 1 modulo 3. So their sum is divisible by 3. ■

Example 1.2 (India RMO 1990 P1)**.** Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any 5 balls of the same colour at least two of them will always be of the same size (radius). Prove that there are at least 3 balls which lie in the same box have the same colour and have the same size (radius).

> **Walkthrough** —
>
> **(a)** At least how many balls are there in the box with more balls?
>
> **(b)** At least how many balls are there in a **large** colour class in this box?
>
> **(c)** What about the sizes of the balls of this colour class?

Solution 2. By the pigeonhole principle, one box contains at least 33 balls. Applying the pigeonhole principle once again, it follows that at least 9 balls in this box are of the same colour. Since among five balls of the same colour, at least two of them are of the same size we conclude that these 9 balls of the same colour are of at most four different sizes. By the pigeonhole principle, at least three of these 9 balls are of the same size. ■

Example 1.3 (India RMO 1996 P7)**.** If A is a fifty-element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that no two numbers from A add up to 100, show that A contains a square.

> **Walkthrough** — Decompose $\{1, 2, \ldots, 100\}$ as the union of the subsets $\{1, 99\}$, $\{2, 98\}$, $\ldots$, $\{49, 51\}$, $\{50\}$, $\{100\}$. Does one of these subsets contain two perfect squares?

Solution 3. Note that the set A is formed by choosing at most one element from each of the following sets $\{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}, \{50\}, \{100\}$. If 100 lies in A, then we are done. Otherwise, A is formed by choosing at most one element from the first 50 sets. Since A has 50 elements, one element of each of these 50 sets lies in A. In particular, one of $36, 64$ lies in A, and hence A contains a perfect square. ■

Example 1.4 (Putnam 2002 A2)**.** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

> **Walkthrough** —
>
> **(a)** Draw a great circle passing through at least two of the five points.
>
> **(b)** At least one closed hemisphere contains at least two of the remaining three points.
>
> **(c)** Conclude!

Solution 4. Draw a great circle passing through at least two of the five points. Then at least one closed hemisphere contains at least two of the remaining three points. This proves the result. See [**AN10**, Example 3.2]. ■

Example 1.5 (India BMath 2006)**.** Show that the maximum number of non-attacking bishops that can be put in an $n \times n$ chessboard is $2n - 2$.

> **Walkthrough** — Does considering a diagonal and the lines parallel to it help? See Fig. 2.

Solution 5. The maximum number of non-attacking bishops that can be put in an $n \times n$ chessboard cannot exceed the number of diagonals (see Figure

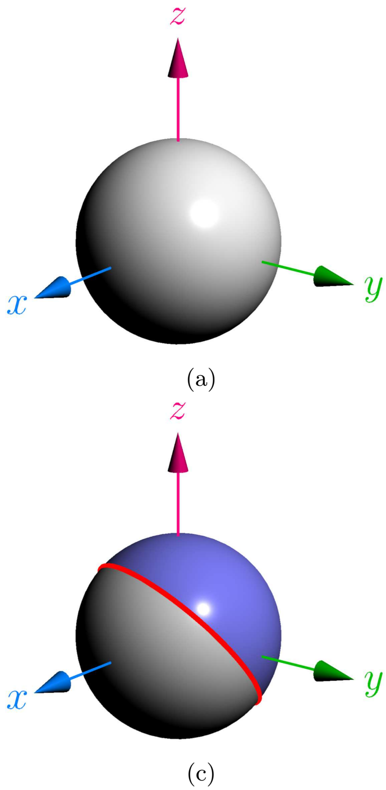

Figure 1: USA Putnam 2002 A2, Example 1.4

2) since each square of the chessboard passes through at least one diagonal and each diagonal can contain at most one bishop. Moreover, the maximum number of non-attacking bishops is one less than the number of diagonals. Otherwise, each diagonal has to accommodate at least one bishop and then the bishops at the top-right corner and at the bottom-left corner would be attacking. If we put bishops at all the squares along the left column and the top row except the top-right square, then these bishops would be non-attacking. Hence the maximum number of non-attacking bishops that can be put in an $n \times n$ chessboard is equal to one less than the total number of diagonals. Hence the required number is equal to $(2n-1)-1 = 2n-2$. ■

Example 1.6 (India RMO 2011a P2)**.** Let $(a_1, a_2, a_3, \dots, a_{2011})$ be a permutation of the numbers $1, 2, 3, \dots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.

Solution 6. Since $a_1, a_2, \dots, a_{2011}$ is a permutation of $1, 2, \dots, 2011$, the sum

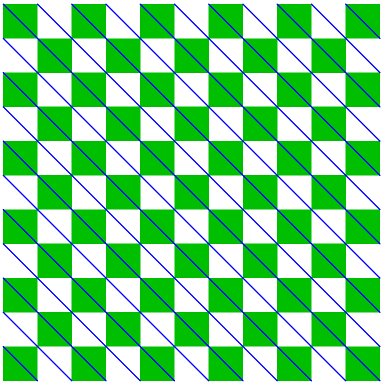

Figure 2: India BMath 2006

$\sum_{j=1}^{2011}(a_j - j)$ is equal to zero, which gives

$$\sum_{\substack{1\le j\le 2011\\ a_j - j>0}} (a_j - j) = - \sum_{\substack{1\le j\le 2011\\ a_j - j<0}} (a_j - j). \quad (1)$$

For any $1 \le j \le 2011$, note that $-2010 \le a_j - j \le 2010$, that is, $0 \le |a_j - j| \le 2010$ holds. If $|a_j - j|, |a_k - k|$ are equal for no j, k with $1 \le j < k \le 2011$, then $|a_1 - 1|, |a_2 - 2|, \ldots, |a_{2011} - 2011|$ are equal to $0, 1, 2, \ldots, 2010$ in some order, and hence

$$|a_1 - 1| + |a_2 - 2| + \cdots + |a_{2011} - 2011| = 1 + 2 + \cdots + 2010 = 1005 \cdot 2011,$$

which is odd and thus contradicts Eq. (1). So there exist two numbers j, k such that $1 \le j < k \le 2011$ and $|a_j - j| = |a_k - k|$. ■

Example 1.7 (India RMO 2013d P6)**.** Suppose that the vertices of a regular polygon of 20 sides are coloured with three colors red, blue and green, such that there are exactly three red vertices. Prove that there are three vertices A, B, C of the polygon having the same colour such that triangle ABC is isosceles.

> **Walkthrough** — Decompose the set of vertices of the 20-gon using the vertices of the pentagons as in Fig. 3.

Solution 7. Since there are exactly three red vertices and any of the remaining 17 vertices are blue or green, it follows that at least 9 of these 17 vertices are of the same color, say blue. Note that the set of vertices of a regular 20-gon can be written as the union of the four pairwise disjoint sets, each of them consisting of the vertices of a regular pentagon. Since there are nine blue vertices, by the

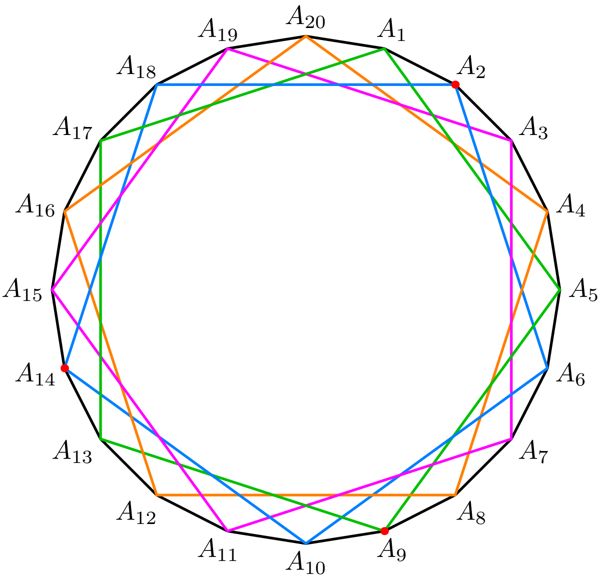

Figure 3: India RMO 2013, Example 1.7

pigeonhole principle, at least one of these four sets contains three blue points. Since any three points on a pentagon form an isosceles triangle, the statement follows. ■

Example 1.8. [**AE11**, Problem 3.10] Show that the numbers 1 to 81 cannot be arranged in a 9×9 chessboard so that the product of the entries of the i-th row is equal to the product of the entries of the i-th column for any i, $1 \leq i \leq 9$.

> **Walkthrough** — Show that the diagonal contains the primes larger than $\frac{81}{2}$. How many primes are there between $\frac{81}{2}$ and 81?

Solution 8. Note that 41 occurs in certain row, say at the i-th row. So 41 divides the product of the entries of the i-th column. Since no number between 1 and 81 is a multiple of 41 except itself, 41 is common to i-th row and the i-th column, that is, it appears on the diagonal. Similarly, the diagonal contains all the primes less than 81 which are larger than 40, that is, it contains $41, 43, 47, 53, 59, 61, 67, 71, 73, 79$. However, 10 primes cannot be put along the diagonal. This proves the result. ■

Example 1.9 (India RMO 2014a P4)**.** Is it possible to write the numbers $17, 18, 19, \ldots, 32$ in a 4×4 grid of unit squares with one number in each square

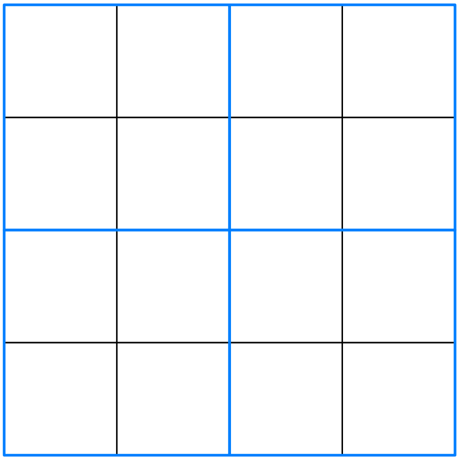

Figure 4: India RMO 2014, Example 1.9

such that if the grid is divided into four 2×2 subgrids of unit squares, then the product of numbers in each of the subgrids divisible by 16?

Walkthrough —

(a) Show that the product of the entries in some subgrid is divisible by 32.

(b) Conclude that the product of all the 16 entries is divisible by $16 \times 16 \times 16 \times 32$.

(c) Is the product of the integers $17, 18, \ldots, 32$ divisible by $16 \times 16 \times 16 \times 32$?

Solution 9. The highest exponents of 2 dividing 32! and 16! are given by

$$v_2(32!) = 16 + 8 + 4 + 2 + 1, \quad v_2(16!) = 8 + 4 + 2 + 1.$$

So the highest power of 2 dividing the product of $17, 18, 19, \ldots, 32$ is 2^{16}. Now if it were possible to write these numbers in a 4×4 grid in the above-mentioned manner, then the product of the numbers in each of the subgrids with blue boundary (see Fig. 4) would be divisible by 2^4. Note that one such subgrid would contain 32, which implies that the product of $17, 18, \ldots, 32$ is divisible by $2^4 \cdot 2^4 \cdot 2^4 \cdot 32 = 2^{17}$, which is impossible. Hence it is not possible to write the integers $17, 18, \ldots, 32$ in a 4×4 grid satisfying the given conditions. ■

Example 1.10 (India INMO 2016 P4). Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

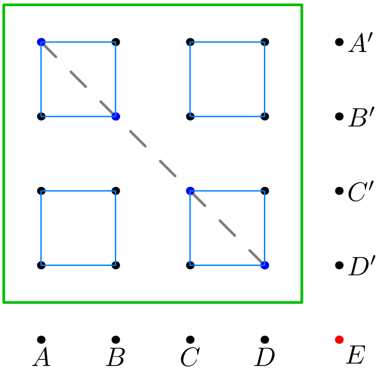

Figure 5: India RMO 2018, Example 1.11

Solution 10. Draw a regular n-gon circumscribed in the circle. If all its vertices are blue, then we are done. Otherwise, we rotate the regular n-gon by a sufficiently small angle so that all its vertices become red. This is possible since there are only finitely many blue points on the circle. ■

Example 1.11 (India RMO 2018a P4)**.** Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \leq m \leq 5, 1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form $(a, b), (a + k, b), (a + k, b + k), (a, b + k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)

Walkthrough —

(a) Assume that the conclusion is false, and that there are more red points than the blue ones.

(b) Show that one of the four corners is red, and assume that the point E is red.

(c) Considering the number of red points in each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$, prove that the green square contains exactly 8 red points, and each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$ contains exactly one red point.

(d) Conclude that the four points lying on the dashed diagonal are blue.

(e) Use the points within the green square, lying outside the diagonal, to form suitable pairs, and show that these points are all red.

Solution 11. On the contrary, let us assume that there is no axes-parallel square having at least three vertices of the same color.

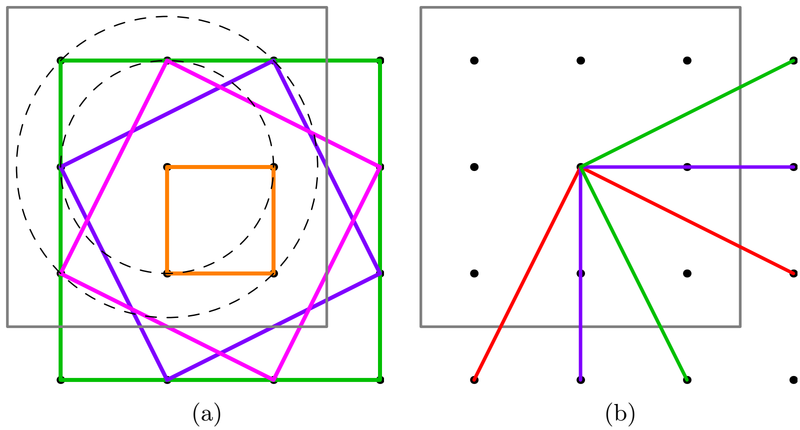

Figure 6: India RMO 2023, Example 1.12

Note that at least 13 among those 25 points are of the same color. Without loss of generality, assume that those are red. By our hypothesis, it follows that among the four vertices at the corners, there is at least one red vertex. Without loss of generality, let us assume that the bottom-right vertex is red, and denote this vertex by E (as in Fig. 5).

Note that each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ (as in Fig. 5) contains at most one red point, otherwise, we can form an axes-parallel square with at least three vertices of the same color. Consequently, the green square (as in Fig. 5) contains at least $13 - 5 = 8$ red points.

If the green square contains at least 9 red points, then at least one of the four blue squares (as in Fig. 5) contains at least three red points, which is not the case by our hypothesis. Hence, the green square contains exactly 8 red points. Consequently, there are precisely 13 red points all together, and each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ contains exactly one red point. By our hypothesis, it follows that all the points on the gray diagonal (as in Fig. 5) are blue. This shows that all the points within the green square, lying outside the gray diagonal, are red. Hence, there exists an axes-parallel square, all whose vertices are red. ■

Example 1.12 (India RMO 2023b P6)**.** Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

The following is from AoPS, is due to Rohan (Goyal?) as mentioned here by L567.

Walkthrough —

(a) Show that if the small square (as in Fig. 6a) does not contain a blue point, then we are done, and assume that the small square contains at least one blue point.

(b) Rotating the configuration about the center of the small square (if necessary), assume that the top-left vertex of the small square (as in Fig. 6a) is blue.

(c) Prove that the gray square contains at most three blue points. (Consider what happens when one of the dashed circle contains more than one blue point.) Conclude that there are at most three blue points within the gray square, as in Fig. 6b.

(d) It suffices to consider that each one of the red, purple, and green L-shapes, has at most one end-point which is blue (otherwise, we are done).

(e) Consider the point at the bottom-right corner, and argue.

Solution 12. Note that the 16 points are the vertices of the four squares. If no vertex of the small square is blue, then by the pigeonhole principle, at least three of the vertices of at least one of the remaining three squares are blue (as in Fig. 6a), and hence there exists an isosceles right-angled triangle with blue vertices.

Let us assume that the small square (as in Fig. 6a) contains a blue point. Rotating the configuration about the center of the small square (if necessary), we may and do assume that the top-left vertex of the small square is blue.

Claim — The gray square (as in Fig. 6a) contains at most three blue points.

Proof of the Claim. We consider the case when at least two blue points lie on at least one of the dashed circles, as in Fig. 6a.

If at least two blue points lie on the bigger dashed circle, then note that no more blue points lies on it, and hence these two blue points lie along a diameter. It follows that no blue point lies on the smaller dashed circle.

If at least two blue points lie on the smaller dashed circle, then using a similar argument, it follows that no blue point lies on the bigger dashed circle.

The Claim follows. □

Note that if both the end-points of one of the red, purple, and green L-shapes (as in Fig. 6b) are blue, then these points together with the center of gray square form the vertices of an isosceles triangle, as required.

Thus, it remains to consider the case when each one of these three L-shapes has at most one end-point which is blue. Applying the above Claim, it follows that the bottom-right point is blue. Note that the center of the gray square, the bottom-right point, and a blue end-point of an L-shape, form the vertices of an isosceles triangle having the required properties. ■

Example 1.13. Given five integers, show that the sum of some three of them is divisible by 3.

Solution 13. If each of the integers $0, 1, 2$ is congruent to at least one of the given integers, then we could take three integers among the five given integers, which are congruent to $0, 1, 2$ modulo 3, respectively, and note that their sum is divisible by 3.

Otherwise, there is a two-element subset A of $\{0, 1, 2\}$ such that each of those five integers is congruent to one of the elements of A modulo 3. Since A contains two elements, by the pigeonhole principle, it follows that at least three of the given five integers are congruent to one of the elements in A. The sum of these three integers is divisible by 3. ■

Example 1.14 (India RMO 2024a P6)**.** Let X be a set of 11 integers. Prove that one can find a nonempty subset $\{a_1, a_2, \dots, a_k\}$ of X such that 3 divides k and 9 divides the sum $\sum_{i=1}^{k} 4^i a_i$.

> **Walkthrough** — Use repeatedly the fact that given five integers, the sum of some five of them is divisible by 3.

Solution 14. From the eleven elements of X, one can find three elements whose sum is divisible by 3. From the remaining 8 elements, one can find three elements whose sum is divisible by 3. From the remaining 5 elements, one can find three elements whose sum is divisible by 3. Hence, there are nine elements in X, denoted by $a_1, a_2, \dots, a_9$ such that the sums

$$a_1 + a_2 + a_3, a_4 + a_5 + a_6, a_7 + a_8 + a_9$$

are divisible by 3. It follows that the sums

$$\alpha := \sum_{i=1}^{3} 4^i a_i, \beta := \sum_{i=4}^{6} 4^i a_i, \gamma := \sum_{i=7}^{9} 4^i a_i$$

are divisible by 3. Each of α, β, γ is congruent to one of $0, 3, 6$ modulo 9. To conclude, let us consider the following cases.

Suppose one of α, β, γ is divisible by 9. Noting that $4^3 \equiv 1 \pmod 9$, we can find a subset of X of size 3 with the desired property. Henceforth, let us assume that none of α, β, γ is divisible by 9.

If all of α, β, γ are congruent to 3 modulo 9, then we can find a subset of X of size 9 with the desired property.

If one of α, β, γ is congruent to 6 modulo 9, and the remaining two are congruent to 3 modulo 9, then we can find a subset of X of size 6 with the desired property.

If one of α, β, γ is congruent to 3 modulo 9, and the remaining two are congruent to 6 modulo 9, then we can find a subset of X of size 6 with the desired property.

If all of α, β, γ are congruent to 6 modulo 9, then we can find a subset of X of size 9 with the desired property.

The completes the proof. ■

Exercise 1.15. [Mat24, Problem 8, §6.4] Show that given 17 integers, the sum of some 9 of them is divisible by 9.

> **Walkthrough** — Use repeatedly the fact that given five integers, the sum of some five of them is divisible by 3.

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