Optimization problems

MOPSS

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Optimization problems

Exercise 1.1 (Benelux Mathematical Olympiad 2020 P2, AoPS)**.** Let N be a positive integer. A collection of $4N^2$ unit tiles with two segments drawn on them as shown is assembled into a $2N \times 2N$ board. Tiles can be rotated.

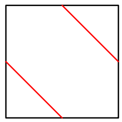


Figure 1: BxMO 2020 P2, Exercise 1.1

The segments on the tiles define paths on the board. Determine the least possible number and the largest possible number of such paths.

For example, there are 9 *paths on the* 4×4 *board shown below.*

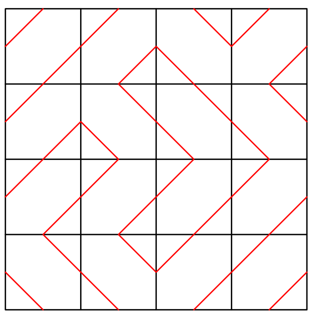
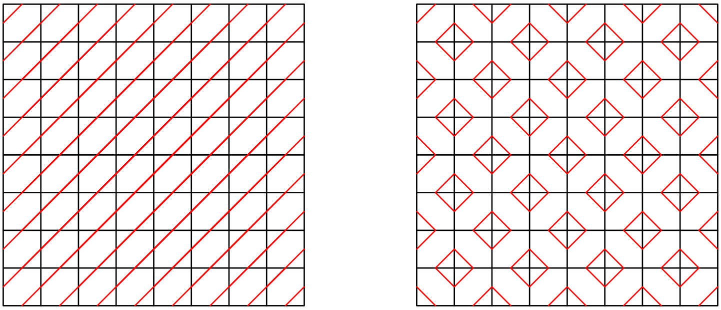


Figure 2: BxMO 2020 P2, Exercise 1.1

Walkthrough —

(a)

Solution 1. Consider a given tiling of a $2N \times 2N$ board using tiles of the given type. Note that there are two types of paths. Some path are between two points lying on the boundary of the square, and the other paths are contained in the interior of the square, and form cycles. Let $\mathcal{B}, \mathcal{C}$ denote the collection of paths of the first and second kind respectively. Observe that among the mid-points of the sides of these $4N^2$ tiles, there are precisely $4 \times 2N = 8N$



(a) (b)

Figure 3: BxMO 2020 P2, Exercise 1.1

mid-points which lie on the boundary of the square. Since each of the paths, which are not cycles, have precisely two such points as end-points, and no two paths of these kind have an end-point in common, it follows that there are precisely $4N$ paths, which are not cycles.

Observe that no two paths have a segment in common. Note that the remaining paths, which are contained in the interior of the square, form cycles. Each of these cycles contains at least four segments. Hence, these cycles together contain at least $4|\mathcal{C}|$ segments. Also note that a path contains at least two segments, unless it is formed by single segment, and note that there are at least four such paths. Let s denote the number of paths consisting of a single segment. Since there are a total of $4N^2$ tiles, each contributing two segments, we have $8N^2$ segments in total. This shows that

$$8N^2 \geq s + 2(|\mathcal{B}| - s) + 4|\mathcal{C}|,$$

which yields

$$8N^2 + 8N \geq 4(|\mathcal{B}| + |\mathcal{C}|) - s.$$

It follows that

$$|\mathcal{B}| + |\mathcal{C}| \leq 2N^2 + 2N + \frac{s}{4} \leq 2N^2 + 2N + 1.$$

This proves that there are exactly $4N$ paths which are not cycles, and there are at most $2N^2 + 2N + 1$ paths in total.

The tiling, as illustrated in Figure 3a in the case $N = 4$, shows an example of a configuration where the number of paths is equal to $4N$. The tiling, as illustrated in Figure 3b in the case $N = 4$, shows an example of a configuration where the number of paths is equal to $2N^2 + 2N + 1$.

Hence, the least possible number and the maximum possible number of paths are $4N$ and $2N^2 + 2N + 1$ respectively. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)