Induction

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Induction

Example 1.1. At time 0, a particle is at the point 0 on the real line. At time 1, the particle divides into two and instantaneously after division, one particle moves 1 unit to the left and the other moves one unit to the right. At time 2, each of these particles divides into two, and one of the two new particles moves one unit to the left and the other moves one unit to the right. Whenever two particles meet, they destroy each other leaving nothing behind. How many particles will be there after time $2^{11} + 1$?

> **Walkthrough** — Try to figure out the number of particles at time $0, 1, 2, 3, 4, \dots$. Do you observe some pattern? If not, determine the number of particles at time t for enough (preferably, consecutive) values of t.

Solution 1. Let us first prove the following claim.

> **Claim** — For any $n \geq 1$, the particles present at time $2^n - 1$ lie at $-(2^n - 1), -(2^n - 3), \dots, (2^n - 3), 2^n - 1$, and until this time no particle was present outside the interval $[-(2^n - 1), (2^n - 1)]$.

Proof of the Claim. Note that this claim holds for $n = 1$. Suppose the claim holds for $n = k$. Then at time 2^k, there are two particles on the real line, present at $-2^k, 2^k$. By the induction hypothesis, the particles present at time $2^k + 2^k - 1 = 2^{k+1} - 1$ lie at $-(2^{k+1} - 1), -(2^{k+1} - 3), \dots, (2^{k+1} - 3), 2^{k+1} - 1$ and until this time no particle was present outside the interval $[-(2^{k+1} - 1), (2^{k+1} - 1)]$. This proves the claim. □

By the above Claim, at time 2^n with $n \geq 1$, there will be precisely two particles, and they will be at $-2^n, 2^n$. Consequently, there will be 4 particles at time $2^{11} + 1$. ■

Example 1.2 (India RMO 1991 P4)**.** There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

1. remove an equal number of balls simultaneously from the urns, and
2. double the number of balls in any one of them.

Show that after performing these operations finitely many times, both the urns can be made empty.

Walkthrough —

(a) If both the urns have an equal number of balls, then removing all the balls from both the urns, we are done.

(b) Otherwise, remove enough balls from both the urns such that one of them is left with only one ball, and the other has one or more balls.

(c) Double the number of balls in the urn containing only one ball, and then go back to the first step.

Solution 2. Denote the urns by A, B and denote the number of balls in them by $n(A), n(B)$ respectively. Without loss of generality, assume that $n(A) \geq n(B)$. Note that it suffices to consider the case $n(B) = 1$ since if we take out $n(B) - 1$ many balls from both the urns (note that $n(B) \geq 1$ and thus $n(B) - 1 \geq 0$), then $n(A) - n(B) + 1$ many balls are left in urn A and exactly one ball is left in urn B.

We apply induction on $n(A)$ to complete the proof. If $n(A) = 1$, then applying the first operation, both the urns can be made empty. Assume that $k \geq 1$ is a positive integer such that both the urns can be made empty by making a finite number of operations of the stated type whenever one urn contains only one ball, and the other urn contains at most k balls.

Suppose there are two urns, denoted by A and B, and the number of balls in them, denoted $n(A), n(B)$ respectively, satisfy $n(A) = k + 1, n(B) = 1$. Now let us keep on doubling the number of balls in urn B until the number of balls in it is greater than or equal to the number of balls in urn A, that is, we double the number of balls in urn B for r times where r is the least positive integer such that $2^r \geq k + 1$. Let us remove k balls from both the urns, and then only one ball is left in urn A and $2^r - k$ balls are left in urn B. Note that $2^r - k < k$ since r is the least positive integer such that $2^r \geq k + 1$. Thus we are left with two urns, one of them containing one ball, and the other containing at most k balls. By the induction hypothesis, these urns can be made empty performing a finite number of operations of the stated type. ■