Inclusion-exclusion principle

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Inclusion-exclusion principle

Example 1.1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution 1. Let A (resp. B) denote the set of integers not exceeding 1000 that are divisible by 7 (resp. 11). Then the size of $A \cup B$ is equal to

$$\#A + \#B - \#A \cap B = \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{77} \right\rfloor = 142 + 90 - 12 = 220.$$

■

Example 1.2 (India RMO 1993 P8)**.** I have 6 friends and during a vacation I met them during several dinners. I found that I dined with all the 6 exactly on 1 day; with every 5 of them on 2 days; with every 4 of them on 3 days; with every 3 of them on 4 days; with every 2 of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?

Walkthrough —

(a) Denote the friends by $F_1, \dots, F_6$. For $1 \leq i \leq 6$, let D_i denote the set of the dinners where F_i was present.

(b) Show that the total number of dinners is 14.

(c) Let n denote the number of dinners that I had alone.

(d) Note that the size of the set $A_1 \cup \dots \cup A_6$ is equal to $14 - n$.

Solution 2. Denote the friends by $F_1, \dots, F_6$. For $1 \leq i \leq 6$, let D_i denote the set of the dinners where F_i was present. Let n denote the number of dinners that I had alone. Since every friend was present in 7 dinners, and was absent in 7 dinners, it follows that the total number of dinners is 14. Consequently, the size of the set $A_1 \cup \dots \cup A_6$ is equal to $14 - n$. By the inclusion-exclusion principle, we obtain

$$14 - n = \sum_{i=1}^{6} |A_i| - \sum_{1 \leq i_1 < i_2 \leq 6} |A_{i_1} \cap A_{i_2}|$$

$$
\begin{aligned}
&+ \sum_{1\le i_1<i_2<i_3\le 6} |A_{i_1}\cap A_{i_2}\cap A_{i_3}| - \sum_{1\le i_1<i_2<i_3<i_4\le 6} |A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}| \\
&+ \sum_{1\le i_1<i_2<i_3<i_4<i_5\le 6} |A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}| \\
&- |A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap A_{i_4}\cap A_{i_5}\cap A_{i_6}| \\
&= \binom{6}{1}7 - \binom{6}{2}5 + \binom{6}{3}4 - \binom{6}{4}3 + \binom{6}{5}2 - 1 \\
&= 42 - 75 + 80 - 45 + 12 - 1 \\
&= 13.
\end{aligned}
$$

This shows that $n = 1$. In other words, I dined alone only on one day. ■

Example 1.3 (India RMO 2015f P6). From the list of natural numbers $1, 2, 3, \ldots,$ suppose we remove all multiples of 7, 11 and 13.

- At which position in the resulting list does the number 1002 appear?
- What number occurs in the position 3600?

Solution 3. Let S denote the set of all positive integers none of which is divisible by 7, 11 or 13. Note that the sum of $\#\{x \in S \mid x \le 1002\}$ and the number of positive integers ≤ 1002 divisible by 7, 11 or 13 is equal to 1002. Moreover, by the inclusion-exclusion principle, the number of positive integers ≤ 1002 divisible by 7, 11 or 13 is equal to

$$
\left\lfloor \frac{1002}{7} \right\rfloor + \left\lfloor \frac{1002}{11} \right\rfloor + \left\lfloor \frac{1002}{13} \right\rfloor - \left\lfloor \frac{1002}{7\cdot 11} \right\rfloor - \left\lfloor \frac{1002}{11\cdot 13} \right\rfloor - \left\lfloor \frac{1002}{13\cdot 7} \right\rfloor + \left\lfloor \frac{1002}{7\cdot 11\cdot 13} \right\rfloor
$$
$$
= 143 + 91 + 77 - 13 - 7 - 11 + 1 = 281.
$$

Note that 1002 belongs to S and hence it appears at the $1002 - 281 = 721$st position.

Suppose n occurs at the 3600th position. So the number of positive integers $\le n$ divisible by 7, 11 or 13 is equal to $n - 3600$. Using the inclusion-exclusion principle, we obtain

$$
n - 3600 = \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor - \left\lfloor \frac{n}{7\cdot 11} \right\rfloor - \left\lfloor \frac{n}{11\cdot 13} \right\rfloor - \left\lfloor \frac{n}{13\cdot 7} \right\rfloor + \left\lfloor \frac{n}{7\cdot 11\cdot 13} \right\rfloor. \tag{1}
$$

This gives

$$
\begin{aligned}
&\frac{n}{7} - 1 + \frac{n}{11} - 1 + \frac{n}{13} - 1 - \frac{n}{7\cdot 11} - \frac{n}{11\cdot 13} - \frac{n}{13\cdot 7} + \frac{n}{7\cdot 11\cdot 13} - 1 \\
&\le n - 3600 \\
&\le \frac{n}{7} + \frac{n}{11} + \frac{n}{13} - \frac{n}{7\cdot 11} + 1 - \frac{n}{11\cdot 13} + 1 - \frac{n}{13\cdot 7} + 1 + \frac{n}{7\cdot 11\cdot 13},
\end{aligned}
$$

which is equivalent to

$$\begin{aligned}&-4\\&\leq n\left(1-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 7}-\frac{1}{7\cdot 11\cdot 13}\right)-3600\\&\leq 3.\end{aligned}$$

Noting that

$$\left(1-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 7}-\frac{1}{7\cdot 11\cdot 13}\right)=\frac{6\cdot 10\cdot 12}{7\cdot 11\cdot 13},$$

we obtain

$$-4\times\frac{7\cdot 11\cdot 13}{6\cdot 10\cdot 12}\leq n-5\cdot 7\cdot 11\cdot 13\leq 3\times\frac{7\cdot 11\cdot 13}{6\cdot 10\cdot 12},$$

which yields

$$-\frac{7\cdot 11\cdot 13}{180}\leq n-5\cdot 7\cdot 11\cdot 13\leq\frac{7\cdot 11\cdot 13}{240},$$

and consequently, any solution of the Eq. (1) satisfies

$$-5\leq n-5\cdot 7\cdot 11\cdot 13\leq 4.$$

Note that $n=5\cdot 7\cdot 11\cdot 13$ is the unique solution to

$$n\left(1-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 7}-\frac{1}{7\cdot 11\cdot 13}\right)-3600=0,$$

and it is a multiple of the pairwise coprime integers $7, 11, 13$. Hence, it is a solution to Eq. (1). Also note that $5\cdot 7\cdot 11\cdot 13-1$ is also a solution to Eq. (1). Moreover, no integer lying in $[-6+5\cdot 7\cdot 11\cdot 13, 4+5\cdot 7\cdot 11\cdot 13]$, other than $5\cdot 7\cdot 11\cdot 13-1$ and $5\cdot 7\cdot 11\cdot 13$ is a solution to Eq. (1). Observe that $5\cdot 7\cdot 11\cdot 13-1$ lies in S.

We conclude that the number that occurs in the 3600th position is $5\cdot 7\cdot 11\cdot 13-1=5004$. ■

Example 1.4 (India RMO 2019b P5)**.** There is a pack of 27 distinct cards, and each card has three values on it. The first value is a shape from $\{\Delta, \square, \odot\}$; the second value is a letter from $\{A, B, C\}$; and the third value is a number from $\{1, 2, 3\}$. In how many ways can we choose an unordered set of 3 cards from the pack, so that no two of the chosen cards have two matching values. For example we can chose $\{\Delta A1, \Delta B2, \odot C3\}$. But we cannot choose $\{\Delta A1, \square B2, \Delta C1\}$.

Solution 4. Let $\mathcal{A}$ denote the set of ordered tuples (u, v, w) with $u, v, w\in(\mathbb{Z}/3\mathbb{Z})^3$ such that no two among u, v, w have two equal coordinates. For $u\in(\mathbb{Z}/3\mathbb{Z})^3$, let $\mathcal{A}_u$ denote the set of ordered tuples lying in $\mathcal{A}$ with the first coordinate equal to u. Note that $(u, v, w) \mapsto (u, v, w) - (u, u, u)$ defines a bijection between $\mathcal{A}_u$ and $\mathcal{A}_0$. Hence, the number of ways of choosing an unordered set of 3 cards satisfying the given conditions is $\frac{1}{3!} \cdot 27 \cdot |\mathcal{A}_0|$. Note that

$$\begin{aligned}\mathcal{A}_0 = \{(0, v, w) \,|\, v \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle, w \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle,\\ w \notin (v + \langle e_1 \rangle) \cup (v + \langle e_2 \rangle) \cup (v + \langle e_3 \rangle)\},\end{aligned}$$

and hence

$$\begin{aligned}&|\mathcal{A}_0|\\ &= \sum_{v \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle} \left| (\langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle)^c \bigcap ((v + \langle e_1 \rangle) \cup (v + \langle e_2 \rangle) \cup (v + \langle e_3 \rangle))^c \right|.\end{aligned}$$

For any subset S of $(\mathbb{Z}/3\mathbb{Z})^3$ and an element $v \in (\mathbb{Z}/3\mathbb{Z})^3$, we have

$$\begin{aligned}|S^c \cap (v + S)^c| &= 27 - |(S^c \cap (v + S)^c)^c|\\ &= 27 - |S \cup (v + S)|\\ &= 27 - |S| - |v + S| + |S \cap (v + S)|\\ &= 27 - 7 - 7 + |S \cap (v + S)|\\ &= 13 + |S \cap (v + S)|.\end{aligned}$$

Henceforth, S denotes the subset $\langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle$ of $(\mathbb{Z}/3\mathbb{Z})^3$, and v denotes an element of $(\mathbb{Z}/3\mathbb{Z})^3$ lying outside S.

First, let us consider the case when v has exactly two nonzero coordinates. It follows that v lies in exactly one of $\langle e_1 \rangle - \langle e_2 \rangle$, $\langle e_2 \rangle - \langle e_3 \rangle$, $\langle e_3 \rangle - \langle e_1 \rangle$. Observe that the set $S \cap (v + S)$ is equal to the union of $\langle e_i \rangle \cap (v + \langle e_j \rangle)$ for $1 \leq i, j \leq n$. It follows that $S \cap (v + S)$ has size two.

Now, let us consider the case when all coordinates of v are nonzero. Note that $\langle e_i \rangle \cap (v + \langle e_j \rangle)$ is empty for any $1 \leq i, j \leq n$. Hence, so is the set $S \cap (v + S)$.

Note that the number of elements of $(\mathbb{Z}/3\mathbb{Z})^3$ having exactly two nonzero coordinates is $3 \cdot 2 \cdot 2 = 12$, and the number of elements of $(\mathbb{Z}/3\mathbb{Z})^3$ having all coordinates nonzero is $2^3 = 8$. It follows that

$$|\mathcal{A}_0| = 12 \cdot (13 + 2) + 8 \cdot (13 + 0) = 284.$$

Hence, the number of ways of choosing an unordered set of 3 cards satisfying the given conditions is $\frac{1}{3!} \cdot 27 \cdot 284 = 1278$. ∎