Grouping in pairs

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Grouping in pairs

Example 1.1 (India RMO 1994 P7)**.** Find the number of all rational numbers m/n such that

1. $0 < m/n < 1$,
2. m and n are relatively prime,
3. $mn = 25!$.

Walkthrough —

(a) Determine the number of pairs of relatively prime positive integers (m, n) satisfying $mn = 25!$.

(b) Show that such pairs (m, n) with $m > n$, and the pairs with $m < n$ are in one-to-one correspondence.

Solution 1. Note that 25! factors as a product of positive integral powers of the 9 primes $2, 3, 5, 7, 11, 13, 17, 19, 23$. So 25! has 2^9 factorizations as a product of two relatively prime positive integers, that is, there are 2^9 pairs of positive integers (m, n) such that m, n are relatively prime and $mn = 25!$. Certainly, we have either $m > n$ or $m < n$. Moreover, interchanging m, n gives a one-to-one correspondence between the factorizations 25! into relatively prime positive integers m, n with $m > n$ and the factorizations 25! into relatively prime positive integers m, n with $m < n$, that is, the map $(m, n) \mapsto (n, m)$ (that is, interchanging m and n) defines a bijection between the sets

$$\begin{aligned} S = \{(m,n) | m, n &\text{ are relatively prime positive integers} \\ &\text{satisfying } mn = 25! \text{ and } \frac{m}{n} < 1\}, \\ T = \{(m,n) | m, n &\text{ are relatively prime positive integers} \\ &\text{satisfying } mn = 25! \text{ and } \frac{m}{n} > 1\}. \end{aligned}$$

Since $S \cup T$ has 2^9 elements, it follows that S has cardinality $2^8 = 256$. ■

Example 1.2 (Putnam 2002 A3, India INMO 2013 P4)**.** [**AF13**, Exercise 4.8, p. 90] Let n be an integer greater than 1 and let T_n be the number of nonempty subsets S of $\{1, 2, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Walkthrough —

(a) Show that the average of any nonempty subset of $\{1, 2, \ldots, n\}$ lies between 1 and n.

(b) Observe that it suffices to show that for any $1 \leq a \leq n$, the subsets of $\{1, 2, \ldots, n\}$ with average equal to a is an odd number.

(c) Show that for any $1 \leq a \leq n$, the nonempty subsets of $\{1, 2, \ldots, n\}$ having average equal to a and not containing a, are in one-to-one correspondence with the subsets of $\{1, 2, \ldots, n\}$ having average equal to a, and which contain a. (Consider the averages of $\{1, 2, 3\}$ and $\{2\}$.)

Solution 2. Let $\mathcal{T}_n$ denote the set of nonempty subsets of $\{1, 2, \ldots, n\}$ having integer average. Note that the average of the elements of any subset of $\{1, 2, \ldots, n\}$ lies between 1 and n. So if a nonempty subset S of $\{1, 2, \ldots, n\}$ has integer average a and S does not contain a, then $S \cup \{a\}$ is also a nonempty subset of $\{1, 2, \ldots, n\}$ with average a. Hence, for any integer $1 \leq a \leq n$, there is a bijection between the nonempty subsets of X with integer average a that does not contain a and the subsets of X with integer average a that contain a, but not equal to $\{a\}$. So the subsets of $\mathcal{T}_n$ with average equal to a is an odd number for any $1 \leq a \leq n$. Hence $T_n - n$ is even. ■

Example 1.3 (India RMO 2012f P6)**.** Let S be the set $\{1, 2, \ldots, 10\}$. Let A be a subset of S. We arrange the elements of A in increasing order, that is, $A = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$. Define WSUM for this subset as $3(a_1 + a_3 + \ldots) + 2(a_2 + a_4 + \ldots)$ where the first term contains the odd numbered terms and the second the even numbered terms. (For example, if $A = \{2, 5, 7, 8\}$, WSUM is $3(2 + 7) + 2(5 + 8)$.) Find the sum of WSUMs over all the subsets of S. (Assume that WSUM for the null set is 0.)

Walkthrough —

(a) Note that

$$\begin{aligned} \mathrm{WSUM}(\{2,5,7,8\}) &= 3(2+7) + 2(5+8), \\ \mathrm{WSUM}(\{1,2,5,7,8\}) &= 3(1+5+8) + 2(2+7), \end{aligned}$$

which shows that

$$\mathrm{WSUM}(\{2,5,7,8\}) + \mathrm{WSUM}(\{1,2,5,7,8\}) = 3 + 5(2+5+7+8).$$

(b) Also note that the sum of all the elements of a subset of A of $\{2, 3, \ldots, 10\}$, and the sum of all the elements of its complement in $\{2, 3, \ldots, 10\}$, add up to the sum of all the elements of $\{2, 3, \ldots, 10\}$.

Solution 3. Note that the subsets of $\{1, 2, \ldots, 10\}$ can be obtained by considering the subsets of $\{2, 3, \ldots, 10\}$, along with the union of these subsets with

$\{1\}$. Using this observation, we decompose the required sum as follows.

$$\begin{aligned}
\sum_{A\subseteq S} \mathrm{WSUM}(A) &= \sum_{A\subseteq\{2,3,\ldots,10\}} \mathrm{WSUM}(A) + \mathrm{WSUM}(A\cup\{1\}) \\
&= \sum_{A\subseteq\{2,3,\ldots,10\}} \left(3 + 5\sum_{a\in A} a\right) \\
&= 3\cdot 2^9 + 5 \sum_{A\subseteq\{2,3,\ldots,10\}} \sum_{a\in A} a \\
&= 3\cdot 2^9 + \frac{5}{2} \sum_{A\subseteq\{2,3,\ldots,10\}} \left(\sum_{a\in A} a + \sum_{a\in A^c} a\right) \\
&= 3\cdot 2^9 + \frac{5}{2} \sum_{A\subseteq\{2,3,\ldots,10\}} (2+3+\cdots+10) \\
&= 3\cdot 2^9 + \frac{5}{2} \times 54 \times 2^9 \\
&= 138 \times 2^9.
\end{aligned}$$

■

References

[**AF13**] T. Andreescu and Z. Feng. *A Path to Combinatorics for Undergraduates: Counting Strategies*. Birkhäuser Boston, 2013. ISBN: 9780817681548. URL: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 2)