

Generating functions

MOPSS

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Mathematics Olympiad
Problem Solving Sessions

MOPSS

DEPARTMENT OF MATHEMATICS
IISER BHOPAL

<https://jpsaha.github.io/MOTP/MOPSS/>

Suggested readings

- **Evan Chen's**
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- **Evan Chen** discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

1.1	Example	2
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§1 Generating functions

See [Sob13, Chapter 6], [Wil06].

Example 1.1. In how many ways, can we fill a bag with n fruits subject to the following constraints?

1. The number of apples must be even.
2. The number of bananas must be a multiple of 5.
3. There can be at most four oranges.
4. There can be at most one pear.

Solution 1. Note that the required number is equal to the coefficient of x^n in the formal power series $A(x)B(x)O(x)P(x)$ where

$$\begin{aligned}A(x) &= 1 + x^2 + x^4 + x^6 + \dots, \\B(x) &= 1 + x^5 + x^{10} + x^{15} + \dots, \\O(x) &= 1 + x + x^2 + x^3 + x^4, \\P(x) &= 1 + x.\end{aligned}$$

Also note that

$$\begin{aligned}A(x)B(x)O(x)P(x) &= \frac{1}{1-x^2} \frac{1}{1-x^5} \frac{1-x^5}{1-x} \times (1+x) \\&= \frac{1}{(1-x)^2} \\&= 1 + 2x + 3x^2 + 4x^3 + \dots\end{aligned}$$

So the bag can be filled in $n + 1$ ways such that the given conditions hold. ■

References

- [Sob13] PABLO SOBERÓN. *Problem-solving methods in combinatorics*. An approach to olympiad problems. Birkhäuser/Springer Basel AG, Basel, 2013, pp. x+174. ISBN: 978-3-0348-0596-4; 978-3-0348-0597-1. DOI: [10.1007/978-3-0348-0597-1](https://doi.org/10.1007/978-3-0348-0597-1). URL: <http://dx.doi.org/10.1007/978-3-0348-0597-1> (cited p. 2)
- [Wil06] HERBERT S. WILF. *generatingfunctionology*. Third. A K Peters, Ltd., Wellesley, MA, 2006, pp. x+245. ISBN: 978-1-56881-279-3; 1-56881-279-5 (cited p. 2)