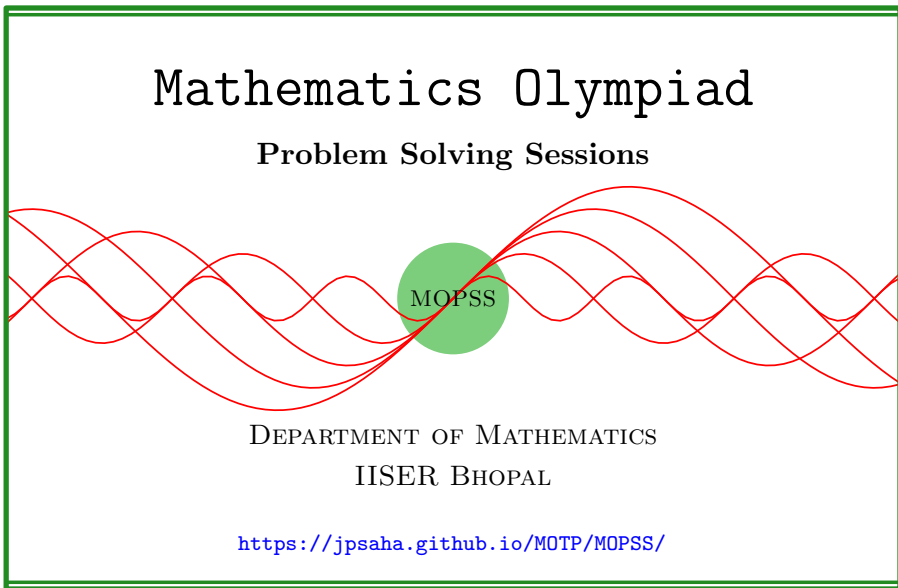


Extremal principle

MOPSS

21 February 2025



Suggested readings

- **Evan Chen's**
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- **Evan Chen** discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

1.1 Example (India RMO 1991 P8) 2

§1 Extremal principle

See [Eng98, Chapter 3].

Example 1.1 (India RMO 1991 P8). The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have 8, 5 or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.

Walkthrough —

- (a) Consider a square containing the maximum (or minimum) of all the entries, denoted by m .
- (b) Show that the entries of the neighbouring squares are equal to m .
- (c) What are the possibilities for the entries of the remaining squares?

Solution 1. Let m denote the maximum among the entries of the 64 squares. Note that if a square contains m , then the entries of its neighbouring squares are equal to m (otherwise, the entry of some of the neighbouring squares would be strictly smaller than m , then the average of the entries of the neighbouring squares would be strictly smaller than m , which is impossible). Consequently, the entries of any two neighbouring squares are equal to m if one of them contains m . Let S denote a square containing m . Note that any other square on the chessboard can be reached from S through a sequence of squares such that the successive squares are neighbours, that is, given any square T other than S , there is a sequence of squares

$$S_1, S_2, \dots, S_n \quad \text{with } S_1 = S, S_n = T$$

such that S_{i+1} is a neighbour of S_i for all $1 \leq i < n$. Then by the above argument, the entries of the squares S_1, S_2, \dots, S_n are equal to m . ■

References

- [Eng98] ARTHUR ENGEL. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited p. 2)