Extremal principle

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Extremal principle

See [**Eng98**, Chapter 3].

Example 1.1 (India RMO 1991 P8)**.** The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have $8, 5$ or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.

Walkthrough —

(a) Consider a square containing the maximum (or minimum) of all the entries, denoted by m.

(b) Show that the entries of the neighbouring squares are equal to m.

(c) What are the possibilities for the entries of the remaining squares?

Solution 1. Let m denote the maximum among the entries of the 64 squares. Note that if a square contains m, then the entries of its neighbouring squares are equal to m (otherwise, the entry of some of the neighbouring squares would be strictly smaller than m, then the average of the entries of the neighbouring squares would be strictly smaller than m, which is impossible). Consequently, the entries of any two neighbouring squares are equal to m if one of them contains m. Let S denote a square containing m. Note that any other square on the chessboard can be reached from S through a sequence of squares such that the successive squares are neighbours, that is, given any square T other than S, there is a sequence of squares

$$S_1, S_2, \ldots, S_n \quad \text{with } S_1 = S, S_n = T$$

such that S_{i+1} is a neighbour of S_i for all $1 \leq i < n$. Then by the above argument, the entries of the squares $S_1, S_2, \ldots, S_n$ are equal to m. ■

References

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. isbn: 0-387-98219-1 (cited p. 2)