Counting in two different ways

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Counting in two different ways

See [**AF13**, §7], [**AE11**, §3.3].

Example 1.1. Find the sum of all distinct four digit numbers that can be formed using the digits $1, 2, 3, 4, 5$, each digit appearing at most once.

Solution 1. Let a, b, c, d be four distinct elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then among the four digit numbers that can be formed using a, b, c, d with no repeated digits, there are exactly six numbers with i in its j's place for any $i \in \{a, b, c, d\}, j \in \{1, 10, 100, 1000\}$. So the sum of such numbers is equal to $(a + b + c + d) \times (6 + 60 + 600 + 6000) = 6666(a + b + c + d)$. This shows that the required sum is equal to

$$\sum_{\{a,b,c,d\} \subseteq \{1,2,\dots,9\}} 6666(a + b + c + d).$$

Note that

$$\begin{aligned} \sum_{\substack{A \subseteq \{1,2,\dots,9\} \\ |A|=4}} \sum_{a \in A} a &= \sum_{a=1}^{9} \sum_{\substack{A \subseteq \{1,2,\dots,9\} \\ |A|=4, a \in A}} a \\ &= \sum_{a=1}^{9} \binom{8}{3} a \\ &= 45 \times \binom{8}{3} \\ &= 25200. \end{aligned}$$

Hence, the required sum is equal to

$$6666 \times 25200.$$

■

Example 1.2 (India BStat-BMath 2014)**.** A class has 100 students. Let a_i, $1 \le i \le 100$, denote the number of friends the i-th student has in the class. For each $0 \le j \le 99$, let c_j denote the number of students having at least j friends. Show that

$$a_1 + a_2 + \cdots + a_{100} = c_1 + c_2 + \cdots + c_{99}.$$

Solution 2. For $1 \leq i \leq 100$, denote the i-th student by s_i. For $1 \leq j \leq 99$, let C_j denote the set of students having at least j friends. Note that for any $1 \leq i \leq 100$,

$$a_i = \sum_{j=1}^{99} 1_{C_j}(s_i)$$

holds, where for $1 \leq j \leq 99$, 1_{C_j} denotes the map, defined on $\{s_1, s_2, \ldots, s_{100}\}$, given by

$$1_{C_j}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ lies in } C_j, \\ 0 & \text{otherwise.} \end{cases}$$

Summing over $1 \leq i \leq 100$, and interchanging the order of summation, we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{100} &= \sum_{j=1}^{99} \sum_{i=1}^{100} 1_{C_j}(s_i) \\ &= \sum_{j=1}^{99} |\{s_i \mid s_i \in C_j\}| \\ &= \sum_{j=1}^{99} c_j. \end{aligned}$$

This completes the proof. ■

Remark 1. The following is **somewhat naive**, and **does require** additional explanation to be included (at which step(s)?). The following explains (with some effort from readers' end, of course!) why the stated result should hold, and it may also help to arrive at the above solution. However, the following lacks some details.

Walkthrough —

$$\begin{array}{cccccccc}
 & c_1 \text{ bullets} & c_2 \text{ bullets} & c_3 \text{ bullets} & \cdots & c_j \text{ bullets} & \cdots & \\
 & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
s_1 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_1 \text{ bullets} \\
s_2 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_2 \text{ bullets} \\
s_3 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_3 \text{ bullets} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_i & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_i \text{ bullets} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_{100} & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_{100} \text{ bullets}
\end{array}$$

Denote the students by $s_1, s_2, \ldots, s_{100}$ and write $s_1, s_2, \ldots, s_{100}$ in a vertical column. Then for each $1 \leq i \leq 100$, put a_i bullets next to s_i (as shown above). Then the sum $a_1 + \cdots + a_{100}$ is equal to the total number of bullets. It turns out that the number of bullets in the j-th column is equal to c_j for any $1 \leq j \leq 99$, proving that the total number of bullets is also equal to $c_1 + c_2 + c_3 + \cdots + c_{99}$.

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. isbn: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)

[**AF13**] T. Andreescu and Z. Feng. *A Path to Combinatorics for Undergraduates: Counting Strategies*. Birkhäuser Boston, 2013. isbn: 9780817681548. url: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 2)