Counting the complement

MOPSS

24 February 2025

Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

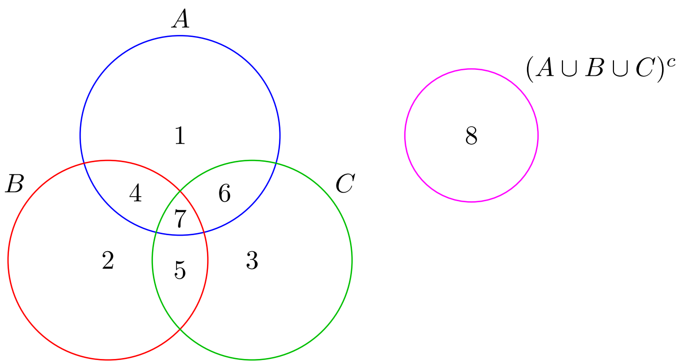

Figure 1: India RMO 2004, Example 1.1

List of problems and examples

§1 Counting the complement

Example 1.1 (India RMO 2004 P4)**.** Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \cap C = \emptyset$, $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$ is $7^n - 2 \cdot 6^n + 5^n$.

> **Walkthrough** — Establish a convenient bijection between the triples of subsets of $\{1, 2, \ldots, n\}$, and the maps from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 8\}$. (Hint: Use Venn diagram.) Use this bijection to count the number of triples satisfying the given conditions.

Solution 1. Note that the triples (A, B, C) of subsets of $\{1, 2, \ldots, n\}$ are in one-to-one correspondense with the maps from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 8\}$. One such correspondense is given by sending (A, B, C) to the map $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 8\}$, which takes the values $1, 2, \ldots, 8$ at the following subsets

$$A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B),$$
$$(A \cap B) \setminus (A \cap B \cap C), (B \cap C) \setminus (A \cap B \cap C), (C \cap A) \setminus (A \cap B \cap C),$$
$$\{1, 2, \ldots, n\} \setminus (A \cup B \cup C),$$

of $\{1, 2, \ldots, n\}$ respectively. Hence, the number of triples (A, B, C) satisfying the given conditions is equal to the number of maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 8\}$, satisfying

$$f^{-1}(7) = \emptyset, f^{-1}(4) \neq \emptyset, f^{-1}(5) \neq \emptyset.$$

Let $\mathcal{F}$ denote the set of maps $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, 8\}$. Applying the inclusion-exclusion principle, we obtain

$$\begin{aligned}
&|\{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset, f^{-1}(4) \neq \emptyset, f^{-1}(5) \neq \emptyset\}| \\
&= \left| \left(\{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset, f^{-1}(4) = \emptyset\} \cup \{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset, f^{-1}(5) = \emptyset\}\right)^c \right| \\
&= |\{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset\}| \\
&\quad - |\{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset, f^{-1}(4) = \emptyset\}| \\
&\quad - |\{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset, f^{-1}(5) = \emptyset\}| \\
&\quad + |\{f \in \mathcal{F} \,|\, f^{-1}(7) = \emptyset, f^{-1}(4) = \emptyset, f^{-1}(5) = \emptyset\}| \\
&= 7^n - 2 \cdot 6^n + 5^n,
\end{aligned}$$

where in the above, for a subset $\mathcal{E}$ of $\mathcal{F}$, the complement of $\mathcal{E}$ in $\mathcal{F}$ is denoted by $\mathcal{E}^c$. This completes the proof. ■

Example 1.2 (India RMO 2009 P4)**.** Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

Walkthrough —

(a) Note that the 3-digit natural numbers containing at least one odd digit, and at least one even digit, form the set $S \setminus (E \cup O)$, where

$$\begin{aligned}
S &= \text{the set of 3-digit integers,} \\
E &= \text{the set of 3-digit numbers having even digits only,} \\
O &= \text{the set of 3-digit numbers having odd digits only.}
\end{aligned}$$

(b) Apply the inclusion-exclusion principle.

Solution 2. Let E (resp. O) denote the set of 3-digit numbers having even (resp. odd) digits only. Denote by S the set of 3-digit integers. Note that the 3-digit numbers with at least one even digit and at least one odd digit form the set $S \setminus (E \cup O)$. By the inclusion-exclusion principle, the size of the set $S \setminus (E \cup O)$ is equal to

$$\sum_{s \in S} s - \sum_{x \in E} x - \sum_{y \in O} y.$$

Note that

$$\sum_{s \in S} s = 10 + 101 + \cdots + 999 = 1099 \cdot 450.$$

Observe that there are 5^3 elements on O. Moreover, for any $1 \leq k \leq 3$ and for any $d \in \{1, 3, 5, 7, 9\}$, there are 5^2 elements of O that contain d in the k-th digit. It follows that

$$\sum_{y \in O} y = 100 \cdot 5^2(1 + 3 + 5 + 7 + 9) + 10 \cdot 5^2(1 + 3 + 5 + 7 + 9)$$

$$\begin{aligned}&+ 5^2(1+3+5+7+9)\\ &= 111 \cdot 25 \cdot 25.\end{aligned}$$

Similarly, there are 5^3 integers of the form $100a + 10b + c$ with $a, b, c \in \{0, 2, 4, 6, 8\}$, and their sum is equal to

$$\begin{aligned}&100 \cdot 5^2(0+2+4+6+8) + 10 \cdot 5^2(0+2+4+6+8) + 5^2(0+2+4+6+8)\\ &= 111 \cdot 25 \cdot 20.\end{aligned}$$

Note that there are 5^2 integers of the form $10b + c$ with $b, c \in \{0, 2, 4, 6, 8\}$, and their sum is equal to

$$10 \cdot 5(0+2+4+6+8) + 5(0+2+4+6+8) = 11 \cdot 5 \cdot 20.$$

This shows that

$$\sum_{x \in E} x = 111 \cdot 25 \cdot 20 - 11 \cdot 5 \cdot 20 = 54400.$$

It follows that

$$\begin{aligned}\sum_{s \in S} s - \sum_{x \in E} x - \sum_{y \in O} y &= 1099 \cdot 450 - 111 \cdot 25 \cdot 25 - 54400\\ &= 495000 - 450 - 62500 - 6250 - 625 - 54400\\ &= 495000 - 450 - 123775\\ &= 370775.\end{aligned}$$

■