Counting via bijections

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

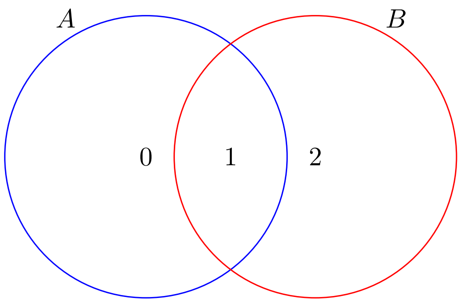

Figure 1: India RMO 1997, Example 1.1

List of problems and examples

§1 Counting via bijections

Example 1.1 (India RMO 1997 P6)**.** Find the number of unordered pairs $\{A, B\}$ (that is, the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n-element set X which satisfy the conditions:

(a) $A \neq B$,

(b) $A \cup B = X$.

(eg., if $X = \{a, b, c, d\}$, then $\{\{a, b\}, \{b, c, d\}\}$, $\{\{a\}, \{b, c, d\}\}$, $\{\emptyset, \{a, b, c, d\}\}$ are some of the admissible pairs.)

Walkthrough — Establish a suitable one-to-one correspondence between the set of the ordered pairs (A, B) with $A \cup B = X$ and the set of maps $f : X \to \{0, 1, 2\}$.

Solution 1. Note that the ordered pairs (A, B) with $A \cup B = X$ are in a one-to-one correspondence with the maps $f : X \to \{0, 1, 2\}$. Such a one-to-one correspondence is given by sending such a pair (A, B) to $f : X \to \{0, 1, 2\}$ taking the values $0, 1, 2$ at the subsets $A \setminus (A \cap B)$, $A \cap B$, $B \setminus (A \cap B)$ of X, respectively (see Fig. 1). This shows that there are 3^n ordered pairs (A, B)

satisfying $A \cup B = X$. Among these pairs, there is only one pair satisfying $A = B$, namely, the pair (X, X). Hence, the number of unordered pairs $\{A, B\}$ satisfying $A \neq B$ and $A \cup B = X$ is equal to $\frac{1}{2}(3^n - 1)$. ■

Example 1.2. Let $S = \{1, 2, \ldots, n\}$. Find the number of unordered pairs $\{A, B\}$ of subsets of S such that A and B are disjoint, where A or B or both may be empty.

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of the ordered pairs (A, B) of subsets of S, and the set of maps $f : X \to \{0, 1, 2\}$.

Solution 2. Note that the number of ordered pairs (A, B) of disjoint subsets of S is equal to the number of functions from S to $\{0, 1, 2\}$, which is equal to 3^n. From such an ordered pair (A, B), we get an unordered pair $\{A, B\}$ as desired. Moreover, an unordered pair $\{A, B\}$ of disjoint subsets of S, comes from (A, B), and also from (B, A). Note that for such an unordered pair $\{A, B\}$, the order pairs $(A, B), (B, A)$ are equal if and only if $A = B = \emptyset$. Hence, the number of the unordered pairs satisfying the given conditions is equal to $1 + \frac{1}{2}(3^n - 1) = \frac{1}{2}(3^n + 1)$. ■

Example 1.3 (India RMO 2012a P4, India RMO 2012b P4, India RMO 2012c P4, India RMO 2012d P4)**.**

1. Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{2, 3, 5, 7\}$.
2. Let $X = \{1, 2, 3, \ldots, 12\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{2, 3, 5, 7, 8\}$.
3. Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{5, 7, 8\}$.
4. Let $X = \{1, 2, 3, \ldots, 11\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 10\}$ satisfying $A \cap B = \{2, 3, 5, 7\}$, and the set of maps $f : \{1, 2, \ldots, 10\} \setminus \{2, 3, 5, 7\} \to \{0, 1, 2\}$.

Solution 3. Note that given a subset Y of a set X, the ordered pairs (A, B) of subsets of X with $A \cap B = Y$ are in one-to-one correspondence with the maps from $X \setminus Y$ to $\{0, 1, 2\}$ (where the inverse images of $0, 1, 2$ correspond to $A \setminus (A \cap B), B \setminus (A \cap B), X \setminus (A \cup B)$, respectively). Moreover, if (A, B) is such an order pair, then $(A, B) = (B, A)$ holds if and only if $A = B = Y$. It follows that the number of ordered pairs $\{A, B\}$ satisfying $A \cap B = Y$ and $A \neq B$ is equal to $\frac{1}{2}(3^{|X \setminus Y|} - 1)$ if $X \setminus Y$ is finite.

The number of pairs of the given type are $3^6 - 1, 3^7 - 1, 3^7 - 1, 3^5 - 1$. ■

Example 1.4 (India RMO 2013b P6). For a natural number n, let $T(n)$ denote the number of ways we can place n objects of weights $1, 2, \dots, n$ on a balance such that the sum of the weights in each pan is the same. Prove that $T(100) > T(99)$.

Walkthrough —

(a) Try to find out a placing of the weights $1, 2, \dots, 99$ on a balance such that the weights on the pans are the same.

(b) How about putting $1, 2, 3, \dots, 49$ on one pan, and $51, 52, \dots, 99$ on the other? Note that the weights on the pans are the same. However, the object of weight 50 has not been included. This can be resolved by placing the weights $(\{1, 2, \dots, 49\} \setminus \{25\}) \cup \{50\}$ on one pan, the weights $\{25\} \cup \{51, 52, \dots, 99\}$ on the other.

(c) Does the above help in finding an injective map from the set of all possible placements of the weights $1, 2, \dots, 99$ on a balance satisfying the required condition, to the set of all possible placements of the weights $1, 2, \dots, 100$ on a balance satisfying the required condition? Can one also have in addition that this map is not surjective?

Solution 4. Let S_n denote the set of pairs (A, B) where A, B are disjoint subsets of $\{1, 2, \dots, n\}$ such that their union is $\{1, 2, \dots, n\}$ and the sum of elements of A is equal to the sum of elements of B. To establish $T(99) < T(100)$, it suffices to construct an injective map $f : S_{99} \to S_{100}$ which is not surjective. Define f by

$$f(A, B) = \begin{cases} (A \cup \{100\} \setminus \{50\}, B \cup \{50\}) & \text{if } 50 \in A, \\ (A \cup \{50\}, B \cup \{100\} \setminus \{50\}) & \text{if } 50 \in B. \end{cases}$$

Note that f is well-defined (since given an element (A, B) of S_{99}, either A contains 50 or B contains 50).

Claim — The map f is injective.

Proof of the Claim. Let $(A, B), (C, D)$ be two elements of S_{99} having the same image under f.

Let us consider the case that A contains 50. Since the second coordinate of $f(C, D) = f(A, B)$ contains 50, it follows that C contains 50. Considering the second coordinate of $f(A, B)$ and that of $f(C, D)$, we obtain $B \cup \{50\} = D \cup \{50\}$, which yields $B = D$ (since $50 \notin B$ and $50 \notin D$). This also gives $A = C$.

If B contains 50, then a similar argument shows that $A = C$, and this gives that $B = D$. □

Claim — The map $f : S_{99} \to S_{100}$ is not surjective.

Proof of the Claim. To show that f is not surjective, it suffices to find a pair (A, B) of subsets of $\{1, 2, \ldots, 100\}$ such that $A \cap B = \emptyset$, $A \cup B = \{1, 2, \ldots, 100\}$, and A contains 50 and 100.

Note that for the sum of the elements of any of the following 50 sets

$$\{1, 100\}, \{2, 99\}, \{3, 98\}, \ldots, \{50, 51\}$$

is equal to 101. So the pair (A, B) belongs to S_{100}, where

$$\begin{aligned} A &= \{1, 100\} \cup \{2, 99\} \cup \cdots \cup \{24, 77\} \cup \{50, 51\}, \\ B &= \{26, 75\} \cup \{27, 74\} \cup \cdots \cup \{49, 52\} \cup \{25, 76\}. \end{aligned}$$

Moreover, the pair (A, B) does not belong to $f(S_{99})$. □

Using the above Claims, it follows that $T(99) < T(100)$. ■

Example 1.5 (India RMO 2015c P6)**.** Let $S = \{1, 2, \ldots, n\}$ and let T be the set of all ordered triples of subsets of S, say (A_1, A_2, A_3), such that $A_1 \cup A_2 \cup A_3 = S$. Determine in terms of n,

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3|,$$

where $|X|$ denotes the number of elements in the set X.

Walkthrough — Find a suitable one-to-one correspondence between the triples of the subsets of S, whose union is S and intersection is a given subset X of S, and the maps from $S \setminus X$ to $\{1, 2, \ldots, 6\}$.

Solution 5. Note that for $0 \le r \le n$, the set S has $\binom{n}{r}$ many subsets with r elements. Also note that for $0 \le r < n$, and given a subset X of S with r elements, the number of ordered tuples (A_1, A_2, A_3) of subsets of S with $A_1 \cap A_2 \cap A_3 = X$ and $A_1 \cup A_2 \cup A_3 = S$ is equal to the number of maps from $S \setminus X$ to $\{1, 2, 3, 4, 5, 6\}$, which is equal to 6^{n-r}. Moreover, there is only one ordered tuples (A_1, A_2, A_3) of subsets of S with $A_1 \cap A_2 \cap A_3 = S$. So

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3| = \sum_{r=0}^{n-1} \binom{n}{r} 6^{n-r} r + n$$

$$= \sum_{r=1}^{n} \binom{n}{r} 6^{n-r} r$$
$$= n \sum_{r=0}^{n-1} \binom{n-1}{r} 6^{n-1-r}$$
$$= n7^{n-1},$$

where the final equality is obtained by applying the binomial theorem, and the second last equality is obtained by counting the size of the following set in two different ways.

$$\{(A, a, f) \mid A \subseteq S, a \in A, f \text{ is a map from } S \setminus A \text{ to } \{1, 2, 3, 4, 5, 6\}\}$$

■

Example 1.6 (India RMO 2024b P6)**.** Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \ldots, a_k$ of integers an *n-chain* if $1 = a_1 < a_2 < \cdots < a_k = n$, a_i divides a_{i+1} for all i, $1 \leq i \leq k-1$. Let $f(n)$ be the number of n-chains where $n \geq 2$. For example, $f(4) = 2$ corresponds to the 4-chains $\{1, 4\}$ and $\{1, 2, 4\}$. Prove that $f(2^m \cdot 3) = 2^{m-1}(m+2)$ for every positive integer m.

Walkthrough —

(a) Let us determine $f(2), f(4), f(8), \ldots$.

- Note that $f(2) = 1$ since $\{1, 2\}$ is the only 2-chain.
- Note that $f(2^2) = 2$ since $\{1, 2^2\}$, $\{1, 2, 2^2\}$ are the only 2^2-chains.
- Note that $f(2^3) = 2^2$ since $\{1, 2^3\}$, $\{1, 2, 2^3\}$, $\{1, 2^2, 2^3\}$, $\{1, 2, 2^2, 2^3\}$ are the only 2^3-chains.

The above examples suggest that $f(2^m) = 2^{m-1}$ for any integer $m \geq 1$. However, this does require a proof, which we will do in a while.

(b) Let us determine $f(2 \cdot 3), f(2^2 \cdot 3), f(2^3 \cdot 3), \ldots$.

- Note that $f(2 \cdot 3) = 3$ since there are only 3 many $2 \cdot 3$-chains.

  Here is the $2 \cdot 3$-chain containing $\mathbf{3}$ as the smallest multiple of 3.
  - $\{1, \mathbf{3}, 2 \cdot 3\}$,

  Here are the $2 \cdot 3$-chains containing $\mathbf{2 \cdot 3}$ as the smallest multiple of 3.
  - $\{1, 2, \mathbf{2 \cdot 3}\}$,
  - $\{1, \mathbf{2 \cdot 3}\}$
- Note that $f(2^2 \cdot 3) = 2^1 \cdot (2+2)$ since there are $2^1 \cdot (2+2)$ many $2^2 \cdot 3$-chains.

  Here are the $2^2 \cdot 3$-chains containing $\mathbf{3}$ as the smallest multiple of 3.
  - $\{1, \mathbf{3}, 2^2 \cdot 3\}$,

- $\{1, \mathbf{3}, 2 \cdot 3, 2^2 \cdot 3\}$,

  Here are the $2^2 \cdot 3$-chains containing $\mathbf{2 \cdot 3}$ as the smallest multiple of 3.
- $\{1, 2, \mathbf{2 \cdot 3}, 2^2 \cdot 3\}$,
- $\{1, \mathbf{2 \cdot 3}, 2^2 \cdot 3\}$,

  Here are the $2^2 \cdot 3$-chains containing $\mathbf{2^2 \cdot 3}$ as the smallest multiple of 3.
- $\{1, 2^2, \mathbf{2^2 \cdot 3}\}$,
- $\{1, 2, 2^2, \mathbf{2^2 \cdot 3}\}$,
- $\{1, \mathbf{2^2 \cdot 3}\}$,
- $\{1, 2, \mathbf{2^2 \cdot 3}\}$,

• Note that $f(2^3 \cdot 3) = 2^2 \cdot (3 + 2)$ since there are $2^2 \cdot (3 + 2)$ many $2^3 \cdot 3$-chains.

  Here are the $2^3 \cdot 3$-chains containing $\mathbf{3}$ as the smallest multiple of 3.
- $\{1, \mathbf{3}, 2^3 \cdot 3\}$,
- $\{1, \mathbf{3}, 2 \cdot 3, 2^3 \cdot 3\}$,
- $\{1, \mathbf{3}, 2^2 \cdot 3, 2^3 \cdot 3\}$,
- $\{1, \mathbf{3}, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3\}$,

  Here are the $2^3 \cdot 3$-chains containing $\mathbf{2 \cdot 3}$ as the smallest multiple of 3.
- $\{1, 2, \mathbf{2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, 2, \mathbf{2 \cdot 3}, 2^2 \cdot 3, 2^3 \cdot 3\}$,
- $\{1, \mathbf{2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, \mathbf{2 \cdot 3}, 2^2 \cdot 3, 2^3 \cdot 3\}$,

  Here are the $2^3 \cdot 3$-chains containing $\mathbf{2^2 \cdot 3}$ as the smallest multiple of 3.
- $\{1, 2^2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, 2, 2^2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, 2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,

  Here are the $2^3 \cdot 3$-chains containing $\mathbf{2^3 \cdot 3}$ as the smallest multiple of 3.
- $\{1, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2^2, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, 2^2, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2^2, \mathbf{2^3 \cdot 3}\}$,

– $\{1, 2, 2^2, \mathbf{2^3 \cdot 3}\}$,

The above examples suggest that $f(2^m \cdot 3) = 2^{m-1}(m+2)$ for any integer $m \geq 1$. Certainly, this does require a proof, which we will do in a while.

Solution 6.

Claim — For any integer $m \geq 1$, there are precisely 2^{m-1} many 2^m-chains. In other words, $f(2^m) = 2^{m-1}$ for any integer $m \geq 1$.

Proof of the Claim. In a 2^m-chain, the smallest term is 1 and the largest term is 2^m. The remaining terms are some (or none if the chain has only two terms) of the powers of 2 lying between 1 and 2^m. Thus, to form a 2^m-chain, one needs to determine the possibilies for the terms other than the smallest and the largest one. To do so, from the remaining $m-1$ many powers of 2 lying between 1 and 2^m, we can choose as many and arrange them in an increasing order. By the multiplication principle, this can be done in 2^{m-1} ways. This proves the Claim. □

Claim — Let m be a positive integer, and $0 \leq k < m$ be an integer. The number of $2^m \cdot 3$-chains, which contain $2^k \cdot 3$ as the smallest multiple of 3, is 2^{m-1}.

Proof of the Claim. If $k = 0$, then the $2^m \cdot 3$-chains are in one-to-one correspondence with the 2^m-chains, and hence, it follows that $f(2^m \cdot 3) = f(2^m) = 2^{m-1}$.

Let us assume that $1 \leq k < m$. Then it is clear that

$$f(2^m \cdot 3) = f(2^k)f(2^{m-k}) + f(2^k)f(2^{m-k}) = 2f(2^k)f(2^{m-k}).$$

Using the first Claim, it follows that

$$f(2^m \cdot 3) = 2^{m-1}.$$

This completes the proof. □

Claim — Let m be a positive integer. The number of $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3, is 2^m.

Proof. Note that the $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3 and also contain 2^m, are in one-to-one correspondence with the 2^m-chains. Moreover, the $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3 and does not contain 2^m, are also in one-to-one correspondence with the 2^m-chains. It follows that the number of $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3, is $2 \cdot f(2^m) = 2^m$. □

Combining the two Claims above, it follows that

$$f(2^m \cdot 3) = \sum_{0 \le k < m} 2^{m-1} + 2^m = 2^{m-1}(m+2).$$

This completes the proof. ■