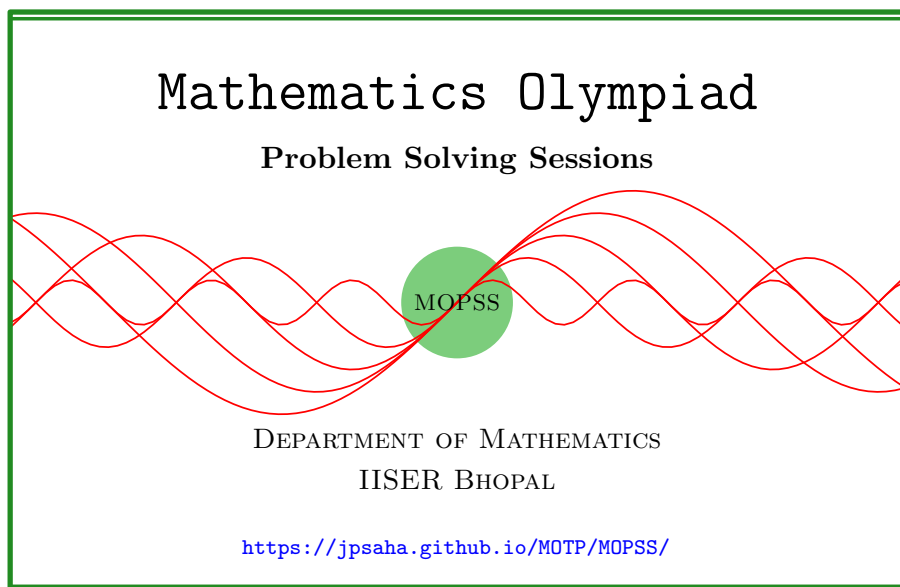


Auxiliary configuration

MOPSS

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Suggested readings

- **Evan Chen's**
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- **Evan Chen** discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

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§1 Auxiliary configuration

Example 1.1 (India RMO 1998 P6). Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T .

Walkthrough — With some trial-and-error, one can find the following example

$$T = \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{b, e, g\}, \{c, d, g\}, \{c, e, f\}\},$$

and then check that it has the desired properties.

Suppose we were not able to make a guess for such a set T ^a. In that situation, an approach would be to assume that such a collection T exists, and then try to find out some additional properties of T **using the hypothesis** that T has the stated properties.

- This approach has the **disadvantage** that the argument would rely on the assumption that such a set T exists, which may not be the case^b.
- However, the **advantage** is that we may be able to make further conclusions about such a putative set T , which may in turn allow us to correctly determine such a set T (or even all such sets T), or to even conclude that no such T exists (of course, depending on the problem).

- Show that the size of T is equal to 7.
- Relabel the elements of A as $1, 2, \dots, 7$.
- Assume that T contains $\{1, 2, 3\}$.
- Show that some of the 3-subsets lying in T , other than $\{1, 2, 3\}$, intersects with $\{1, 2, 3\}$ at exactly one element.
- Reordering the elements of A if necessary, prove that T contains $\{1, 4, 5\}$.
- Next, prove that T contains $\{1, 6, 7\}$, $\{2, 4, 6\}$, $\{3, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 5, 6\}$, and conclude that

$$T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$

- Wait!** We have **only** proved that a putative T is equal to the above up to a reordering of the elements of A . This **does not guarantee** the existence of any collection T having the desired properties.
- Thus, it still remains to find a collection T with the prescribed properties. Does the above collection of 3-subsets of A work?

^aYes, do assume that we were not that clever!

^bThe problem, as stated, indicates the existence of such a T , but this does not suffice.

Solution 1. Suppose T is a set consisting of some size 3-subsets of A such that any size two subset of A is contained in exactly one element of T . Consider the set

$$X = \{(P, Q) \mid P \subseteq Q, |P| = 2, Q \in T\}.$$

Note that X contains precisely $3 \cdot |T|$ many elements. For each subset P of A of size two, there is precisely one element (P, Q) in X . In other words, the map $(P, Q) \mapsto P$ from $X \rightarrow \binom{A}{2}$ is a bijection. It follows that $|T| = \frac{1}{3} \binom{7}{2} = 7$, provided there is a set T with the stated properties.

Note that if such a set T exists and it contains $\{a, b, c\}$, then we claim that some of the remaining 3-subsets lying in T has nonempty intersection with $\{a, b, c\}$. Otherwise, the remaining subsets would be subsets of $A \setminus \{a, b, c\}$, and hence T would contain at most $1 + \binom{7-3}{3} = 1 + 4 < 7 = |T|$ elements. This proves the claim. Hence, some of the 3-subsets lying in T , other than $\{a, b, c\}$, intersects with $\{a, b, c\}$ at exactly one element. ¹

Let's relabel the elements of A as $1, 2, \dots, 7$ for simplicity. The set T contains $\{1, 2, 3\}$ and some of the remaining elements of T intersects with $\{1, 2, 3\}$ at exactly one element. By reordering the elements of A if necessary, we assume that an element of T intersects $\{1, 2, 3\}$ at $\{1\}$. Note that this element contains none of 2, 3. By reordering the elements of A once again if required, we assume that T contains $\{1, 4, 5\}$. Since T contains an element containing $\{1, 6\}$, it follows that $\{1, 6, 7\}$ lies in T . Since T contains an element containing $\{2, 4\}$, it follows that $\{2, 4, 6\}$ or $\{2, 4, 7\}$ lies in T . Reordering 6, 7 if necessary, we assume that T contains $\{2, 4, 6\}$. Since T contains an element containing $\{3, 4\}$, it follows that $\{3, 4, i\}$ lies in T for some $1 \leq i \leq 7$ with $i \notin \{1, 2, 5, 6\}$, i.e., T contains $\{3, 4, 7\}$. Since T contains an element containing $\{2, 5\}$, it follows that $\{2, 5, 7\}$ lies in T . Since T contains an element containing $\{3, 5\}$, it follows that $\{3, 5, 6\}$ lies in T . Using that $|T| = 7$, it follows that

$$T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$

Note that the above collection does have the desired properties. ■

¹Having proved this statement, one may realize that it is much easier to establish! Indeed, such a collection T is nonempty, and hence contains the set $\{a, b, c\}$ up to a reordering of the elements of A . By hypothesis, some element of T contains $\{a, d\}$, and that element intersects with $\{a, b, c\}$ at precisely one element. This argument can be used to replace the longer argument above. It should be noted that the above argument introduces a **crucial idea**, namely, to determine the number of certain objects (here the number of elements of T), it is often helpful to take a detour by counting the number of objects of another type (here the number of pairs of the form (P, Q) satisfying suitable conditions). A similar idea is discussed in ??.

Remark. The above argument also shows that up to a reordering of the elements of A , such a set T is equal to

$$\{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$

Example 1.2 (India RMO 2023a P4). The set X of N four-digit numbers formed from the digits 1, 2, 3, 4, 5, 6, 7, 8 satisfies the following condition: *for any two different digits from 1, 2, 3, 4, 5, 6, 7, 8, there exists a number in X which contains both of them.* Determine the smallest possible value of N .

First, let's work on it. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X , we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1, \dots, 8\}$ which does not occur as a digit in x . Note that after this modification of an element of X , the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1, \dots, 8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X , it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1, 2, \dots, 8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1, \dots, 8\}$ such that any size two subsets of $\{1, \dots, 8\}$ is contained in some element of X .

Consider the set

$$A := \{(P, Q) \mid |P| = 2, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $\binom{4}{2}|X| = 6|X|$. By hypothesis, it follows that A contains at least $\binom{8}{2} = 28$ elements. Hence, X contains at least 5 elements.

With some effort, one can find the following set

$$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{1, 5, 2, 6\}, \{3, 7, 4, 8\}, \{1, 7, 8, 2\}, \{3, 5, 6, 4\}\}.$$

So far, we have proved that X contains at least 5 elements. Based on this information and the above example only (possibly together with the fact that we are not able to come up with a set consisting of five size 4 subsets of $\{1, \dots, 8\}$ having the stated property), we are not in a position to conclude that X has cardinality 6.

In fact, a modification of the above argument does prove that $|X| \geq 6$. Indeed, consider the set

$$A' := \{(P, Q) \mid |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1, \dots, 8\}$, it is contained in at least three elements of X . Indeed, there are 7 size two subsets

of $\{1, \dots, 8\}$ containing i , and the union of no two size 4 subsets of $\{1, \dots, 8\}$ contains all the size two subsets of $\{1, \dots, 8\}$ containing i . By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1, \dots, 8\}$, it is contained in at least three elements of X . Indeed, there are 7 size two subsets of $\{1, \dots, 8\}$ containing i , and the union of no two size 4 subsets of $\{1, \dots, 8\}$ contains all the size two subsets of $\{1, \dots, 8\}$ containing i . By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements. ♣

Solution 2. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X , we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1, \dots, 8\}$ which does not occur as a digit in x . Note that after this modification of an element of X , the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1, \dots, 8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X , it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1, 2, \dots, 8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1, \dots, 8\}$ such that any size two subsets of $\{1, \dots, 8\}$ is contained in some element of X .

Consider the set

$$A := \{(P, Q) \mid |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $4|X|$. Also note that for any element $i \in \{1, \dots, 8\}$, it is contained in at least three elements of X . Indeed, there are 7 size two subsets of $\{1, \dots, 8\}$ containing i , and the union of no two size 4 subsets of $\{1, \dots, 8\}$ contains all the size two subsets of $\{1, \dots, 8\}$ containing i . By hypothesis, it follows that A contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

Note that the set

$$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{1, 5, 2, 6\}, \{3, 7, 4, 8\}, \{1, 7, 8, 2\}, \{3, 5, 6, 4\}\}$$

has size 6 and it consists of certain size 4 subsets of $\{1, \dots, 8\}$ such that any size two subset of $\{1, \dots, 8\}$ is contained in one such size 4 subset.

This proves that the smallest possible value of N is 6. ■