Algebra

Regional Mathematical Olympiad Problems

1990–2025

§1 Warm up

Exercise 1 (RMO 2015e P3,). Find all fractions which can be written simultaneously in the forms

$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6\ell-1}{4\ell-3}$$

for some integers k, ℓ.

Exercise 2 (RMO 2016e P5,).

(i) A 7-tuple $(a_1, a_2, a_3, a_4, b_1, b_2, b_3)$ of pairwise distinct positive integers with no common factor is called a *shy* tuple if

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = b_1^2 + b_2^2 + b_3^2$$

and for all $1 \le i < j \le 4$ and $1 \le k \le 3$, $a_i^2 + a_j^2 \ne b_k^2$. Prove that there exist infinitely many shy tuples.

(ii) Show that 2016 can be written as a sum of squares of four distinct natural numbers.

Exercise 3 (RMO 2019a P4,). Consider the following 3×2 array formed by using the numbers $1, 2, 3, 4, 5, 6$,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the square of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers $7, 8, 9, \ldots, 3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \quad \text{and} \quad \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2$$

Exercise 4 (RMO 2025b P2,). Let a, b, c be distinct nonzero real numbers satisfying

$$a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a}.$$

Determine the value of $|a^2b + b^2c + c^2a|$.

§2 $a^3 + b^3 + c^3 - 3abc$

Exercise 5 (RMO 2002 P2,). Solve the following equation for real x:

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

§3 Polynomials

Exercise 6 (RMO 2017a P3,). Let $P(x) = x^2 + \frac{x}{2} + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all real roots of $P(Q(x)) = 0$.

§4 Quadratic polynomials

Exercise 7 (RMO 2003 P6,). Find all real numbers a for which the equation

$$x^2 + (a - 2)x + 1 = 3|x|$$

has exactly three distinct real solutions for x.

Exercise 8 (RMO 2007 P3,). Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

Exercise 9 (RMO 2010 P2,). Let

$$P_1(x) = ax^2 - bx - c, P_2(x) = bx^2 - cx - a, P_3(x) = cx^2 - ax - b$$

be three quadratic polynomials where a, b, c are nonzero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

Exercise 10 (RMO 2012f P1,). Find nonzero real numbers a, b such that $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are three distinct polynomials with a common root.

Exercise 11 (RMO 2013c P4,). A polynomial is called a *Fermat polynomial* if it can be written as the sum of squares of two polynomials with integer coefficients. Suppose that $f(x)$ is a Fermat polynomial such that $f(0) = 1000$. Prove that $f(x) + 2x$ is not a Fermat polynomial.

Exercise 12 (RMO 2015a P2,). Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a, b are real numbers. Suppose $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$ is an AP of positive integers. Prove that a, b are integers.

Exercise 13 (RMO 2015b P2,). Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a is real and $b \neq 2$, is rational. Suppose $P(0)^2, P(1)^2, P(2)^2$ are integers, prove that a and b are integers.

Exercise 14 (RMO 2015d P2,). Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numbers s is not equal to t such that $P(s) = t$ and $P(t) = s$. Prove that $b - st$ is a root of the equation $x^2 + ax + b - st = 0$.

Exercise 15 (RMO 2015e P2,). Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that $P_1(m) = P_2(n), P_2(m) = P_1(n)$. Prove that $a_1 - a_2$ is even.

Exercise 16 (RMO 2023a P3,). Let $f(x)$ be a polynomial with real coefficients of degree 2. Suppose that for some pairwise distinct **nonzero** real numbers , a, b, c we have:

$$f(a) = bc, f(b) = ac, f(c) = ab.$$

Dertermine $f(a + b + c)$ in terms of a, b, c.

Exercise 17 (RMO 2024a P2,). Show that there do not exist non-zero real numbers a, b, c such that the following statements hold simultaneously:

1. the equation $ax^2 + bx + c = 0$ has two distinct roots x_1, x_2;
2. the equation $bx^2 + cx + a = 0$ has two distinct roots x_2, x_3;
3. the equation $cx^2 + ax + b = 0$ has two distinct roots x_3, x_1.

(Note that x_1, x_2, x_3 may be real or complex numbers.)

§5 Cubic polynomials

Exercise 18 (RMO 1999 P4,). If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, show that $p = q = r$.

Exercise 19 (RMO 2012b P6,). Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Exercise 20 (RMO 2013a P2,). Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:

(a) $f(1) = 0$,

(b) the roots of $g(x) = 0$ are the squares of the roots of $f(x) = 0$.

Find the value of $a^{2013} + b^{2013} + c^{2013}$.

Exercise 21 (RMO 2013e P6,). Let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$, where a and b are nonzero real numbers. Suppose that the roots of the equation $P(x) = 0$ are the reciprocals of the roots of the equation $Q(x) = 0$. Prove that a and b are integers. Find the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$.

Exercise 22 (RMO 2016f P3,). Find all integers k such that all roots of the following polynomial are also integers:

$$f(x) = x^3 - (k-3)x^2 - 11x + (4k-8).$$

§6 Quartics

Exercise 23 (RMO 2012a P6,). Let a and b be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

§7 Viète's relations

Exercise 24 (RMO 2012e P2,). cf. [GA17, Problem 141] Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

§8 AP, GP, HP

Exercise 25 (RMO 1994 P1,). A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf?

Exercise 26 (RMO 2009 P6,). In a book with page numbers from 1 to 100 some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

Exercise 27 (RMO 2011b P3,). Let $a, b, c > 0$. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and if $a^2 + b^2$, $b^2 + c^2$, $c^2 + a^2$ are in geometric progression, show that $a = b = c$.

Exercise 28 (RMO 2014a P2,). Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let

$$\sum_{i=1}^{n} a_{2i-1}^2 = x, \quad \sum_{i=1}^{n} a_{2i}^2 = y, \quad a_n + a_{n+1} = z.$$

Express d in terms of x, y, z, n.

Exercise 29 (RMO 2016a P6,). Let $\langle a_1, a_2, a_3, \ldots \rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

Exercise 30 (RMO 2016b P6,). Show that the infinite arithmetic progression $\langle 1, 4, 7, 10, \ldots \rangle$ has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

Exercise 31 (RMO 2016c P6,).

(a) Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.

(b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

Exercise 32 (RMO 2016d P6,).

(i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.

(ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

Exercise 33 (RMO 2018b P2,). Find the set of all real values of a for which the real polynomial equation $P(x) = x^2 - 2ax + b = 0$ has real roots, given that $P(0) \cdot P(1) \cdot P(2) \neq 0$ and $P(0), P(1), P(2)$ form a geometric progression.

§9 Rational and irrational numbers

Exercise 34 (RMO 2013d P4,). Let x be a nonzero real number such that $x^4 + \frac{1}{x^4}$ and $x^5 + \frac{1}{x^5}$ are both rational numbers. Prove that $x + \frac{1}{x}$ is a rational number.

Exercise 35 (RMO 2019a P1,). Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.

§10 Binomial theorem

Exercise 36 (RMO 1996 P6,). Given any positive integer n, show that there are two positive rational numbers a and b, $a \neq b$, which are not integers and which are such that $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers.

§11 Telescoping

Exercise 37 (RMO 2018b P6,). Define a sequence $\{a_n\}_{n\geq 1}$ of real numbers by

$$a_1 = 2, \qquad a_{n+1} = \frac{a_n^2+1}{2}, \text{ for } n \geq 1.$$

Prove that

$$\sum_{j=1}^{N} \frac{1}{a_j+1} < 1$$

for every natural number N.

§12 Finite differences

Exercise 38 (RMO 2013b P3,). Consider the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2.$$

Prove that there exists a natural number $n > 2013$ for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

§13 On the growth of polynomials

Exercise 39 (RMO 2015c P3,). Let $P(x)$ be a nonconstant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n)-2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

§14 Integer divisibility

Exercise 40 (RMO 2015c P3,). Let $P(x)$ be a nonconstant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n)-2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

Exercise 41 (RMO 2016g P8,). At some integer points a polynomial with integer coefficients take values $1, 2$ and 3. Prove that there exist not more than one integer at which the polynomial is equal to 5.

Exercise 42 (RMO 2025b P6,). Let $p(x)$ be a nonconstant polynomial with integer coefficients, and let $n \geq 2$ be an integer such that no term of the sequence

$$p(0), p(p(0)), p(p(p(0))), \ldots$$

is divisible by n. Show that there exist integers a, b such that $0 \leq a < b \leq n-1$ and n divides $p(b) - p(a)$.

§15 Functional equations

Exercise 43 (RMO 2006 P7,). Let X be the set of all positive integers greater than or equal to 8, and let $f : X \to X$ be a function such that $f(x+y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

§16 System of equations

Exercise 44 (RMO 1992 P8,). Solve the system

$$\begin{aligned}(x+y)(x+y+z) &= 18\\ (y+z)(x+y+z) &= 30\\ (z+x)(x+y+z) &= 2A\end{aligned}$$

in terms of the parameter A.

Exercise 45 (RMO 1993 P7,). In a group of ten persons, each person is asked to write the sum of the ages of all the other 9 persons. If all the ten sums form the 9-element set $\{82, 83, 84, 85, 87, 89, 90, 91, 92\}$, find the individual ages of the persons (assuming them to be whole numbers of years).

Exercise 46 (RMO 1994 P4, , All-Russian MO 1991 Grade 10 Day 2 P7). Solve the system of equations for real x and y:

$$5x\left(1 + \frac{1}{x^2+y^2}\right) = 12, \quad 5y\left(1 - \frac{1}{x^2+y^2}\right) = 4.$$

Exercise 47 (RMO 1996 P3,). Solve for real number x and y:

$$xy^2 = 15x^2 + 17xy + 15y^2, \quad x^2y = 20x^2 + 3y^2.$$

Exercise 48 (RMO 1997 P3,). Solve for real x:

$$\frac{1}{[x]} + \frac{1}{[2x]} = x - [x] + \frac{1}{3},$$

where $[x]$ is the greatest integer less than or equal to x (eg, $[3.4] = 3$ and $[-0.8] = -1$).

Exercise 49 (RMO 2000 P7,). Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

Exercise 50 (RMO 2001 P3,). Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z, eg, $[\frac{7}{4}] = 1$).

Exercise 51 (RMO 2002 P7,). Find all integers a, b, c, d satisfying the following relations:

1. $1 \leq a \leq b \leq c \leq d$;
2. $ab + cd = a + b + c + d + 3$.

Exercise 52 (RMO 2003 P4,). Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:

$$x \leq y \leq z, x + y + z \leq 100.$$

Exercise 53 (RMO 2004 P2,). Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.

Exercise 54 (RMO 2011a P6,). Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

Exercise 55 (RMO 2012d P6,). Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1,$$
$$\frac{1}{yz} = \frac{y}{x} + 1,$$
$$\frac{1}{zx} = \frac{z}{y} + 1.$$

Exercise 56 (RMO 2012f P3,). Solve

$$2^{5x+2[x]} = 11 \cdot 2^{5x} + 5 \cdot 2^{7[x]}$$

for $x \in \mathbb{R}, \mathbb{Q}$.

Exercise 57 (RMO 2014c P4,). Find all positive reals x, y, z such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2014},$$
$$2y - 2z + \frac{1}{x} = \frac{1}{2014},$$
$$2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

Exercise 58 (RMO 2015a P6,). Show that there are infinitely many positive real numbers a, which are not integers, such that $a(a - 3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Exercise 59 (RMO 2015d P6, , cf. RMO 2015e P6). Find all real numbers a such that $4 < a < 5$ and $a(a-3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Exercise 60 (RMO 2015e P6, , cf. RMO 2015d P6). Find all real numbers a such that $3 < a < 4$ and $a(a-3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Exercise 61 (RMO 2019b P3,). Find all triples of non-negative real numbers (a, b, c) which satisfy the following set of equations

$$a^2 + ab = c,$$
$$b^2 + bc = a,$$
$$c^2 + ca = b.$$

§17 Inequalities

§17.1 Warm up

Exercise 62 (RMO 1990 P2, , Nesbitt's inequality 1903, Moscow Mathematical Olympiad 1963 Grade 9 Second Round P3, UK BMO 1976 P2). Let $a, b, c > 0$. Then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

§17.2 No square is negative ($x^2 \geq 0$)

Exercise 63 (RMO 1995 P7,). Show that for any real number x:

$$x^2 \sin x + x \cos x + x^2 + \frac{1}{2} > 0.$$

Exercise 64 (RMO 2014c P2,). Find all real x, y such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1).$$

Exercise 65 (RMO 2014d P2,). If x and y are positive real numbers, prove that

$$4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy.$$

Exercise 66 (RMO 2015c P7,). Let $x, y, z \in \mathbb{R}$, such that $x^2 + y^2 + z^2 - 2xyz = 1$. Prove that

$$(1 + x)(1 + y)(1 + z) \leq 4 + 4xyz.$$

Exercise 67 (RMO 2015f P1, Putnam 1998 B1). Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x \in \mathbb{R}^+$.

Exercise 68 (RMO 2018a P2,). Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1 + x^{2k}} = \frac{n(n + 1)}{4}.$$

§17.3 Manipulation

Exercise 69 (RMO 2000 P3,). Suppose $(x_1, x_2, \cdots, x_n, \cdots)$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq \cdots$, and for all n,

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all $k \geq 1$ the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \leq 3.$$

Exercise 70 (RMO 2002 P6,). For any natural number $n > 1$, prove the inequality

$$\frac{1}{2} < \frac{1}{n^2 + 1} + \frac{2}{n^2 + 2} + \frac{3}{n^2 + 3} + \cdots + \frac{n}{n^2 + n} < \frac{1}{2} + \frac{1}{2n}.$$

Exercise 71 (RMO 2005 P3,). If a, b, c are three real numbers such that $|a - b| \geq |c|, |b - c| \geq |a|, |c - a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.

§17.4 Rearrangement inequality

Exercise 72 (RMO 2012a P3, , RMO 2012b P3, RMO 2012c P3, RMO 2012d P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \leq 1$.

Exercise 73 (RMO 2012b P3, , RMO 2012a P3, RMO 2012c P3, RMO 2012d P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \leq 1$.

Exercise 74 (RMO 2012c P3, , RMO 2012a P3, RMO 2012b P3, RMO 2012d P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \leq 1$.

Exercise 75 (RMO 2012d P3, , RMO 2012a P3, RMO 2012b P3, RMO 2012c P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \leq 1$.

Exercise 76 (RMO 2014b P2,)**.** Let x, y, z be positive real numbers. Prove that

$$\frac{y^2+z^2}{x} + \frac{z^2+x^2}{y} + \frac{x^2+y^2}{z} \geq 2(x+y+z).$$

Exercise 77 (RMO 2017a P6,)**.** Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}$$

Exercise 78 (RMO 2025a P2,)**.** Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{2a^2}{a^3+1} + \frac{2b^2}{b^3+1} + \frac{2c^2}{c^3+1} \leq a^2 + b^2 + c^2.$$

§17.5 Cauchy–Schwarz inequality

Exercise 79 (RMO 2013d P3,)**.** Given real numbers $a, b, c, d, e > 1$. Prove that

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20.$$

Exercise 80 (RMO 2014a P6,)**.** Let $x_1, x_2, x_3, \ldots, x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq 1. \tag{1}$$

Exercise 81 (RMO 2016d P2,). Let a, b, c be positive real numbers such that

$$\frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$

Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}.$$

We refer to Exercise 104 and Exercise 105.

Exercise 82 (RMO 2023b P5,). Let $n > k > 1$ be positive integers. Determine all positive real numbers $a_1, a_2, \ldots, a_n$ which satisfy

$$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n.$$

§17.6 QM-AM-GM-HM inequality

Exercise 83 (RMO 1991 P2,). If a, b, c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

Exercise 84 (RMO 1993 P6,). If a, b, c, d are four positive real numbers such that $abcd = 1$, prove that

$$(1+a)(1+b)(1+c)(1+d) \geq 16.$$

Exercise 85 (RMO 1994 P8, , All-Russian MO 1991 Grade 11 Day 1 P3). If a, b and c are positive real numbers such that $a + b + c = 1$, prove that

$$(1+a)(1+b)(1+c) \geq 8(1-a)(1-b)(1-c).$$

Exercise 86 (RMO 2003 P3,). Let a, b, c be three positive real numbers such that $a+b+c = 1$. Prove that among the three numbers $a-ab, b-bc, c-ca$, there is one which is at most $\frac{1}{4}$ and there is one which is at least $\frac{2}{9}$.

Exercise 87 (RMO 2005 P7,). Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

Exercise 88 (RMO 2006 P3,). If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq 3.$$

Exercise 89 (RMO 2007 P6,). Prove that

(a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$,

(b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$,

(c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Exercise 90 (RMO 2008 P3,). Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1$ are positive real numbers. Prove that

(i) $0 < 3ab \leq 1$,

(ii) $b \geq \sqrt{3}$.

Exercise 91 (RMO 2011b P6,). Find the largest real constant λ such that

$$\frac{\lambda abc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2 \tag{2}$$

for all positive real numbers a, b, c.

Exercise 92 (RMO 2012e P4,). Let a, b, c be positive real numbers such that $abc(a+b+c) = 3$. Prove that we have

$$(a+b)(b+c)(c+a) \geq 8.$$

Also determine the case of equality.

Exercise 93 (RMO 2012f P8,). Let x, y, z be positive real numbers such that $2(xy + yz + zx) = xyz$. Prove that

$$\frac{1}{(x-2)(y-2)(z-2)} + \frac{8}{(x+2)(y+2)(z+2)} \leq \frac{1}{32}.$$

If a, b, c are positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, then show that

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

Exercise 94 (RMO 2014e P5,). Let a, b, c be positive real numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1. \tag{3}$$

Prove that

$$(1+a^2)(1+b^2)(1+c^2) \geq 125.$$

When does the equality hold?

Exercise 95 (RMO 2016c P2,). Let a, b, c be three distinct positive real numbers such that $abc = 1$. Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

Exercise 96 (RMO 2016e P4,). Let a, b, c be positive real numbers such that $a + b + c = 3$. Determine, with certainty, the largest possible value of the expression

$$\frac{a}{a^3+b^2+c} + \frac{b}{b^3+c^2+a} + \frac{c}{c^3+a^2+b}.$$

Exercise 97 (RMO 2016f P5,). Let x, y, z be non-negative real numbers such that $xyz = 1$. Prove that

$$(x^3+2y)(y^3+2z)(z^3+2x) \geq 27.$$

Exercise 98 (RMO 2019a P3,). Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+a^3+c^3} + \frac{c}{c^2+a^3+b^3} \leq \frac{1}{5abc}.$$

§17.7 Bunching terms

Exercise 99 (RMO 1992 P7,). Show that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} < \frac{4}{3}$$

Exercise 100 (RMO 1998 P3, , Canadian Mathematical Olympiad 1998 P3). Prove the following inequality for every natural number $n \geq 2$:

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

§17.8 Look at the difference

Exercise 101 (RMO 2004 P7,). Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that

1. $y < x < 1$ and
2. $x^2 + y^2 < 1$.

Exercise 102 (RMO 2017b P5). If $a, b, c, d \in \mathbb{R}$ such that $a > b > c > d > 0$ and $a + d = b + c$; then prove that

$$\frac{(a+b)-(c+d)}{\sqrt{2}} > \sqrt{a^2+b^2} - \sqrt{c^2+d^2}.$$

Exercise 103 (RMO 2024b P4,). Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Show that there exist i, j with $1 \leq i < j \leq 4$, such that $(a_i - a_j)^2 \leq \frac{1}{5}$.

§17.9 Algebraic substitutions

Exercise 104 (RMO 2016a P2, , RMO 2016b P2)**.** Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that $abc \leq \frac{1}{8}$.

We refer to Exercise 105.

Exercise 105 (RMO 2016b P2, , RMO 2016a P2)**.** Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that $abc \leq \frac{1}{8}$.

We refer to Exercise 81.

§17.10 Ravi substitution

Exercise 106 (RMO 1996 P5,)**.** Let ABC be a triangle and h_a the altitude through A. Prove that

$$(b+c)^2 \geq a^2 + 4h_a^2.$$

Exercise 107 (RMO 1999 P5,)**.** If a, b, c are the sides of a triangle, prove the following inequality:

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3.$$

Exercise 108 (RMO 2001 P6,)**.** If x, y, z are the sides of a triangle, then prove that

$$|x^2(y-z) + y^2(z-x) + z^2(x-y)| < xyz.$$

§17.11 Triangle inequality

Exercise 109 (RMO 1990 P5,)**.** Let P be a point inside a triangle ABC. Let s denote the semiperimeter $\frac{1}{2}(AB + BC + CA)$. Prove that

$$s < AP + BP + CP < 2s.$$

Exercise 110 (RMO 1995 P5,)**.** Show that for any triangle ABC, the following inequality is true:

$$a^2 + b^2 + c^2 > \sqrt{3}\max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\}.$$

Exercise 111 (RMO 1997 P5,). Let x, y and z be three distinct real positive numbers. Determine with proof whether or not the three real numbers

$$\left|\frac{x}{y}-\frac{y}{x}\right|, \left|\frac{y}{z}-\frac{z}{y}\right|, \left|\frac{z}{x}-\frac{x}{z}\right|$$

can be the lengths of the sides of a triangle.

Exercise 112 (RMO 2009 P5,). A convex polygon is such that the distance between any two vertices does not exceed 1.

(i) Prove that the distance between any two points on the boundary of the polygon does not exceed 1.

(ii) If X and Y are two distinct points inside the polygon, prove that there exists a point Z on the boundary of the polygon such that $XZ + YZ \leq 1$.

Exercise 113 (RMO 2014e P1,). Three positive real numbers a, b, c are such that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. Can a, b, c be the lengths of the sides of a triangle? Justify your answer.

§17.12 Geometric inequalities

Exercise 114 (RMO 1993 P4,). Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD; and similarly r_2 is the radius of the circle passing through B and C and touching AD. Show that

$$r_1 + r_2 \geq \frac{5}{8}(a + b).$$

Exercise 115 (RMO 1997 P4,). In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that

(a) $AD \cdot BC \geq AB \cdot CD$,

(b) $AD + BC \geq AB + CD$.

Exercise 116 (RMO 2013b P5,). Let $n \geq 3$ be a natural number and let P be a polygon with n sides. Let $a_1, a_2, \ldots, a_n$ be the lengths of sides of P and let p be its perimeter. Prove that

$$\frac{a_1}{p-a_1} + \frac{a_2}{p-a_2} + \cdots + \frac{a_n}{p-a_n} < 2.$$

References

[**GA17**] Răzvan Gelca and Titu Andreescu. *Putnam and beyond*. Second. Springer, Cham, 2017, pp. xviii+850. ISBN: 978-3-319-58986-2; 978-3-319-58988-6. DOI: 10.1007/978-3-319-58988-6. URL: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 4)