Viète's relations

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Viète's relations

Example 1.1 (India RMO 2012e P2). cf. [**GA17**, Problem 141] Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Solution 1. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the roots of P. Note that

$$\begin{aligned}
&\sum_{1\leq i,j\leq n, i\neq j} (\alpha_i - \alpha_j)^2 \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 2\sum_{1\leq i,j\leq n, i\neq j} \alpha_i\alpha_j \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2(n-1)\left(\sum_{1\leq i\leq n} \alpha_i\right)^2 - 4(n-1)\left(\sum_{1\leq i<j\leq n} \alpha_i\alpha_j\right) - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\binom{n}{2} \\
&= 0.
\end{aligned}$$

Since $\alpha_1, \dots, \alpha_n$ are real, it follows that they are all equal. Using $\alpha_1 + \cdots + \alpha_n = n$, we get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1.$$

This implies that $a_i = -\binom{n}{i}$ for any $0 \leq i \leq n-1$. ■

Solution 2. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the roots of $P(x)$. Note that

$$\begin{aligned}
&(\alpha_1 - 1)^2 + (\alpha_2 - 1)^2 + \cdots + (\alpha_n - 1)^2 \\
&= \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \\
&= (\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2 - 2\sum_{1\leq i<j\leq n} \alpha_i\alpha_j + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \\
&= n^2 - n(n-1) + n - 2n \\
&= 0.
\end{aligned}$$

So the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all equal to 1. This implies that $P(x) = (x-1)^n$, and hence a_i is equal to $(-1)^i \binom{n}{i}$ for any $0 \leq i \leq n$. ■

References

[**GA17**] Răzvan Gelca and Titu Andreescu. *Putnam and beyond*. Second. Springer, Cham, 2017, pp. xviii+850. isbn: 978-3-319-58986-2; 978-3-319-58988-6. doi: 10.1007/978-3-319-58988-6. url: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 2)