Telescoping

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Telescoping

Example 1.1 (IMOSL 1996 A7, Armenia)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x \in \mathbb{R}$, we have $|f(x)| \leq 1$ and

$$f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right) = f(x) + f\left(x + \frac{13}{42}\right).$$

Show that f is periodic.

Solution 1. Consider the function $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = f(x + 1/7) - f(x)$. Note that $g(x + 1/6) = g(x)$ holds for all $x \in \mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ denote the function, defined by

$$h(x) = g(x) + g\left(x + \frac{1}{7}\right) + g\left(x + \frac{2}{7}\right) + \cdots + g\left(x + \frac{6}{7}\right).$$

Note that $h(x) = f(x + 1) - f(x)$ and $h(x + 1/6) = h(x)$ holds for all $x \in \mathbb{R}$. For any $x \in \mathbb{R}$ and any integer $r \geq 1$, we have

$$f(x + r) - f(x) = h(x) + h(x + 1) + \cdots + h(x + r - 1) = rh(x),$$

and this shows that $|rh(x)| \leq 2$. This implies that h is the zero function, and hence, f is periodic. ■

Example 1.2 (India RMO 2018b P6)**.** Define a sequence $\{a_n\}_{n\geq 1}$ of real numbers by

$$a_1 = 2, \qquad a_{n+1} = \frac{a_n^2 + 1}{2}, \text{ for } n \geq 1.$$

Prove that

$$\sum_{j=1}^{N} \frac{1}{a_j + 1} < 1$$

for every natural number N.

Solution 2. Note that

$$2(a_{n+1} - 1) = a_n^2 - 1$$

holds for any $n \geq 1$. Also note that $a_1 > 1$, and by induction, it follows that $a_n > 1$ for any $n \geq 2$. This shows that

$$\frac{1}{a_n + 1} = \frac{a_n + 1 - 2}{a_n^2 - 1}$$

$$
\begin{aligned}
&= \frac{1}{a_n - 1} - \frac{2}{a_n^2 - 1} \\
&= \frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1}
\end{aligned}
$$

holds for any $n \geq 1$. Consequently, for any natural number N, we obtain

$$
\sum_{j=1}^{N} \frac{1}{a_j + 1} = \sum_{j=1}^{N} \left(\frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1} \right) = \frac{1}{a_1 - 1} - \frac{1}{a_{N+1} - 1} < \frac{1}{a_1 - 1} = 1.
$$

■