System of equations

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 System of equations

We refer to [AG09, §2.5] for further problems.

§1.1 Linear equations

Example 1.1 (India RMO 1993 P7)**.** In a group of ten persons, each person is asked to write the sum of the ages of all the other 9 persons. If all the ten sums form the 9-element set $\{82, 83, 84, 85, 87, 89, 90, 91, 92\}$, find the individual ages of the persons (assuming them to be whole numbers of years).

Solution 1. The ten sums are

$$82, 83, 84, 85, 87, 89, 90, 91, 92, n$$

for a unique $n \in \{82, 83, 84, 85, 87, 89, 90, 91, 92\}$. Denote the individual ages by $a_1, \ldots, a_{10}$. Reordering them if necessary, we have

$$\begin{aligned}
(a_1 + \cdots + a_{10}) - a_1 &= 82,\\
(a_1 + \cdots + a_{10}) - a_2 &= 83,\\
(a_1 + \cdots + a_{10}) - a_3 &= 84,\\
(a_1 + \cdots + a_{10}) - a_4 &= 85,\\
(a_1 + \cdots + a_{10}) - a_5 &= 87,\\
(a_1 + \cdots + a_{10}) - a_6 &= 89,\\
(a_1 + \cdots + a_{10}) - a_7 &= 90,\\
(a_1 + \cdots + a_{10}) - a_8 &= 91,\\
(a_1 + \cdots + a_{10}) - a_9 &= 92,\\
(a_1 + \cdots + a_{10}) - a_{10} &= n.
\end{aligned}$$

Adding them yields

$$9(a_1 + \cdots + a_{10}) = 82 + 83 + 84 + 85 + 87 + 89 + 90 + 91 + 92 + n.$$

Since the integer 9 divides $82 + 83 + 84 + 85 + 87 + 89 + 90 + 91 + 92 + n$, it follows that

$$1 + 2 + 3 + 4 + 6 + 8 + 0 + 1 + 2 + n$$

is divisible by 9, which shows that n is divisible by 9. Among the integers $82, 83, 84, 85, 87, 89, 90, 91, 92$, only 90 is divisible by 9. It follows that $n = 90$. Thus

$$\begin{aligned}
a_1 + \cdots + a_{10} &= \frac{1}{9}(82 + 83 + 84 + 85 + 87 + 89 + 90 + 91 + 92 + 90)\\
&= \frac{1}{9}(10 \times 81 + 1 + 2 + 3 + 4 + 6 + 8 + 9 + 10 + 11 + 9)
\end{aligned}$$

$$= \frac{1}{9}(810 + 63) = 97.$$

This gives

$$a_1 = 15, a_2 = 14, a_3 = 13, a_4 = 12, a_5 = 10,$$
$$a_6 = 8, a_7 = 7, a_8 = 6, a_9 = 5, a_{10} = 7.$$

■

§1.2 Nonlinear equations

Example 1.2 (India RMO 1992 P8)**.** Solve the system

$$\begin{aligned}(x+y)(x+y+z) &= 18\\(y+z)(x+y+z) &= 30\\(z+x)(x+y+z) &= 2A\end{aligned}$$

in terms of the parameter A.

Solution 2. The given system can be rewritten as

$$\begin{aligned}(x+y+z)^2 - z(x+y+z) &= 18\\(x+y+z)^2 - x(x+y+z) &= 30\\(x+y+z)^2 - y(x+y+z) &= 2A.\end{aligned}$$

Adding them, we obtain $(x+y+z)^2 = 24 + A$. Note that

$$\begin{aligned}x(x+y+z) &= (x+y+z)^2 - 30\\&= -6 + A,\\y(x+y+z) &= (x+y+z)^2 - 2A\\&= 24 - A,\\z(x+y+z) &= (x+y+z)^2 - 18\\&= 6 + A\end{aligned}$$

holds. Note that $x + y + z \neq 0$, which gives $A \neq -24$ and hence, we obtain

$$(x, y, z) = \left(\pm\frac{-6 + A}{\sqrt{24 + A}}, \pm\frac{24 - A}{\sqrt{24 + A}}, \pm\frac{6 + A}{\sqrt{24 + A}}\right)$$

where the signs $+$ and $-$ correspond, and $\sqrt{24 + A}$ denotes a square root of $24 + A$ in $\mathbb{C}$. ■

Example 1.3 (Hungary MO 2001/02 Grades 11 and 12 – specialized math classes P1)**.** The real numbers $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ are such that each number equals the sum of the squares of the others. Find all such numbers $x_1, \ldots, x_n$.

Solution 3. The given condition yields

$$\begin{aligned} x_1^2 + x_1 &= x_1^2 + x_2^2 + \cdots + x_n^2, \\ x_2^2 + x_2 &= x_1^2 + x_2^2 + \cdots + x_n^2, \\ &\vdots \\ x_n^2 + x_n &= x_1^2 + x_2^2 + \cdots + x_n^2. \end{aligned}$$

Adding them, it follows that

$$x_1 + x_2 + \cdots + x_n = (n-1)(x_1^2 + x_2^2 + \cdots + x_n^2).$$

Note that $x_i \geq 0$ for any $1 \leq i \leq n$. Moreover, for any $1 \leq i < j < n$,

$$x_i^2 + x_i = x_j^2 + x_j$$

holds, which yields

$$(x_i - x_j)(x_i + x_j + 1) = 0,$$

and using $x_i, x_j \geq 0$, it follows that $x_i = x_j$. Combining $x_1 = x_2 = \cdots = x_n$ with

$$x_1 + x_2 + \cdots + x_n = (n-1)(x_1^2 + x_2^2 + \cdots + x_n^2),$$

we obtain

$$nx_1 = (n-1)nx_1^2,$$

which shows that $x_1 = 0$ or $x_1 = \frac{1}{n-1}$.

Also note that if

$$x_1 = x_2 = \cdots = x_n = 0,$$

then each x_i is the sum of the sqaures of the others. Moreover, if

$$x_1 = x_2 = \cdots = x_n = \frac{1}{n-1},$$

then each x_i is also the sum of the sqaures of the others.

We conclude that there are two solutions, which are

$$x_1 = x_2 = \cdots = x_n = 0$$

and

$$x_1 = x_2 = \cdots = x_n = \frac{1}{n-1}.$$

■

Example 1.4 (India RMO 2004 P2)**.** Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.

Solution 4. Denote the integers on the top and bottom face of the cube by p, q respectively, and the integers written on four other faces by a, b, c, d. So the integers pab, pbc, pcd, pda are written at the four vertices of the face at the top, and the integers qab, qbc, qcd, qda are written at the four vertices of the face at the bottom. The sum of these eight integers is equal to 2004, that is,

$$(p+q)(ab+bc+cd+da) = 2004,$$

which gives

$$(p+q)(a+c)(b+d) = 2^2 \cdot 3 \cdot 167.$$

Note that $2, 3, 167$ are primes. Since a, b, c, d, p, q are positive integers, each of $p+q, a+c, b+d$ is at least 2, and exactly one of them has two prime factors (counting multiplicities). So the integers $p+q, a+c, b+d$ are equal to $2\cdot 2, 3, 167$, or $2 \cdot 3, 2, 167$, or $2 \cdot 167, 2, 3$, or $3 \cdot 167, 2, 2$ in some order. Also note that there are positive integers a, b, c, d, p, q such that $p+q, a+c, b+d$ take these values. Thus the possible values of T are

$$\begin{aligned} 2 \cdot 2 + 3 + 167 &= 174, \\ 2 \cdot 3 + 2 + 167 &= 175, \\ 2 \cdot 167 + 2 + 3 &= 339, \\ 3 \cdot 167 + 2 + 3 &= 506. \end{aligned}$$

■

§1.3 Considering the difference

Example 1.5 (India RMO 2002 P7)**.** Find all integers a, b, c, d satisfying the following relations:

1. $1 \leq a \leq b \leq c \leq d$;
2. $ab + cd = a + b + c + d + 3$.

Solution 5. Let a, b, c, d be integers satisfying the given conditions. Let p, q, r be integers satisfying $b = a + p, c = b + q, d = c + r$. Note that p, q, r are nonnegative. The given conditions yield

$$a(a+p) + (a+p+q)(a+p+q+r) = 4a + 3p + 2q + r + 3,$$

which is equivalent to

$$2a^2 + (3p+2q+r)a + (p+q)(p+q+r) = 4a + 3p + 2q + r + 3 = 0,$$

that is,

$$2a(a-2) + (a-1)(3p+2q+r) + (p+q)(p+q+r) = 3. \tag{1}$$

From Eq. (1), we get $a \leq 2$.

If $a = 1$ holds, then $(p+q)(p+q+r) = 5$, which gives

$$p+q = 1, p+q+r = 5,$$

that is, (p,q,r) is equal to $(1,0,4)$ or $(0,1,4)$.

Let us consider the case that $a = 2$. Note that

$$(3p+2q+r) + (p+q)(p+q+r) = 3,$$

which implies that $p = 0$. It follows that $2q + r + q(q+r) = 3$, which gives $q \leq 1$, and hence (q,r) is equal to $(0,3)$ or $(1,0)$.

The above argument shows that (p,q,r) is equal to one of

$$(1,0,4), (0,1,4), (0,0,3), (0,1,0).$$

Consequently, (a,b,c,d) is equal to one of

$$(1,2,2,6), (1,1,2,6), (2,2,2,5), (2,2,3,3).$$

Moreover, the above tuples also satisfy the given conditions.

We conclude that the above tuples are precisely all the solutions to the given relations. ■

Example 1.6 (India RMO 2003 P4)**.** Find the number of ordered triples (x,y,z) of nonnegative integers satisfying the conditions:

$$x \leq y \leq z, x+y+z \leq 100.$$

Solution 6. Substituting $y = x + a, z = y + b$, we get $3x + 2a + b \leq 100$. So it suffices to find number of solutions of the inequality $2a + b \leq 100 - 3x$ in nonnegative integers. Note that given a nonnegative integer k, the number of solutions of the inequality $2u+v = k$ in nonnegative integers is equal to $\lfloor \frac{k}{2} \rfloor + 1$. Hence given a nonnegative integer k, the number of solutions of $2u + v \leq k$ in nonnegative integers is equal to

$$\sum_{t=0}^{k} \left(\left\lfloor \frac{t}{2} \right\rfloor + 1 \right).$$

Note that if $k = 2\ell + 1$, then

$$\begin{aligned}\sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor &= \sum_{t=0}^{2\ell+1} \left\lfloor \frac{t}{2} \right\rfloor \\ &= 1 + 1 + 2 + 2 + \cdots + \ell + \ell \\ &= 2\,(1 + 2 + \cdots + \ell)\end{aligned}$$

$$= 2\left(1 + 2 + \cdots + \left\lfloor \frac{k}{2} \right\rfloor\right),$$

and if $k = 2\ell$, then

$$\begin{aligned}\sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor &= \sum_{t=0}^{2\ell} \left\lfloor \frac{t}{2} \right\rfloor \\ &= 1 + 1 + 2 + 2 + \cdots + (\ell - 1) + (\ell - 1) + \ell \\ &= 2\left(1 + 2 + \cdots + \ell\right) - \ell \\ &= 2\left(1 + 2 + \cdots + \left\lfloor \frac{k}{2} \right\rfloor\right) - \left\lfloor \frac{k}{2} \right\rfloor.\end{aligned}$$

Thus

$$\sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor = 2\left(1 + 2 + \cdots + \left\lfloor \frac{k}{2} \right\rfloor\right) - \frac{1 + (-1)^k}{2} \left\lfloor \frac{k}{2} \right\rfloor,$$

and hence given a nonnegative integer k, the number of solutions of $2u + v \leq k$ in nonnegative integers is equal to

$$\begin{aligned}\sum_{t=0}^{k} \left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) &= k + 1 + \sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor \\ &= k + 1 + 2\left(1 + 2 + \cdots + \left\lfloor \frac{k}{2} \right\rfloor\right) - \frac{1 + (-1)^k}{2} \left\lfloor \frac{k}{2} \right\rfloor.\end{aligned}$$

So the number of solutions of $2a + b \leq 100 - 3x$ in nonnegative integers is equal to

$$\begin{aligned}&\sum_{x=0}^{33} \sum_{t=0}^{100-3x} \left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) \\ &= \sum_{x=0}^{33} \left(100 - 3x + 1 + 2\left(1 + 2 + \cdots + \left\lfloor \frac{100 - 3x}{2} \right\rfloor\right) - \frac{1 + (-1)^{100-3x}}{2} \left\lfloor \frac{100 - 3x}{2} \right\rfloor\right).\end{aligned}$$

Decomposing this sum over even and odd values of x, we get

$$\begin{aligned}&\sum_{x=0}^{33} \sum_{t=0}^{100-3x} \left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) \\ &= \sum_{m=0}^{16} \left(100 - 6m + 1 + 2\left(1 + 2 + \cdots + \left\lfloor \frac{100 - 6m}{2} \right\rfloor\right) - \left\lfloor \frac{100 - 6m}{2} \right\rfloor\right) \\ &\qquad + \sum_{m=0}^{16} \left(100 - 6m - 3 + 1 + 2\left(1 + 2 + \cdots + \left\lfloor \frac{100 - 6m - 3}{2} \right\rfloor\right)\right) \\ &= \sum_{m=0}^{16} \big((50 - 3m + 1 + 2\left(1 + 2 + \cdots + (50 - 3m)\right)\big)\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(100 - 6m - 2 + 2\left(1 + 2 + \cdots + (48 - 3m)\right)\right)\Big) \\
&= \sum_{m=0}^{16} \left(-(50 - 3m) + 1 + 4\left(1 + 2 + \cdots + (50 - 3m)\right)\right) \\
&= \sum_{m=0}^{16} \left(3m - 49 + 2(50 - 3m)(51 - 3m)\right) \\
&= \sum_{m=0}^{16} (18m^2 - 603m + 5051) \\
&= 3 \cdot 16 \cdot 17 \cdot 33 - 603 \cdot 8 \cdot 17 + 5051 \cdot 17 \\
&= 17(1600 - 16 + 5051 - 4824) = 17 \cdot 1811 = 30787.
\end{aligned}
$$

Consequently, there are 30787 ordered triples (x, y, z) of nonnegative integer satisfying the given conditions. ■

The following is a cleaner and much more streamlined solution to the above problem. It is from a post on AoPS by Rijul Saini. See also this page.

Solution 7. Let us establish the following.

> **Claim —** Let m be a nonnegative integer. The number of solutions of
>
> $$p \leq q, p + q \leq m$$
>
> in nonnegative integers is equal to
>
> $$\begin{cases} \frac{(m+1)(m+3)}{4} & \text{if } m \text{ is odd}, \\ \frac{(m+2)^2}{4} & \text{if } m \text{ is even}. \end{cases}$$

Proof of the Claim. Note that for a nonnegative integer k, the number of solutions of

$$x \leq y, x + y = k,$$

in the nonnegative integers is equal to

$$\begin{cases} \frac{k+1}{2} & \text{if } k \text{ is odd}, \\ \frac{k+2}{2} & \text{if } k \text{ is even}. \end{cases}$$

Also note that the Claim follows for $m = 0$.

If m is an odd positive integer, then the number of solutions of

$$x \leq y, x + y \leq m$$

in the nonnegative integers is equal to

$$1 + \frac{1+1}{2} + \frac{2+2}{2} + \frac{3+1}{2} + \frac{4+2}{2} + \frac{5+1}{2} + \cdots + \frac{(m-1)+2}{2} + \frac{m+1}{2}$$

$$= 1 + 1 + 2 + 2 + 3 + 3 + \cdots + \frac{m+1}{2} + \frac{m+1}{2}$$
$$= \frac{m+1}{2}\frac{m+3}{2}.$$

It follows from the above that if m is an even positive integer, then the number of solutions of

$$x \leq y, x + y \leq m$$

in the nonnegativeg integers is equal to

$$\frac{(m-1)+1}{2}\frac{(m-1)+3}{2} + \frac{m+2}{2} = \frac{(m+2)^2}{4}.$$

This proves the Claim. □

Note that $x + y + z \leq 100$ is equivalent to $(y - x) + (z - x) \leq 100 - 3x$, and if $x \leq y \leq z$, then $y - x \geq 0$ and $z - x \geq 0$. It follows that the solutions of the given equation are in one-to-one correspondence with the solutions of the equation $Y + Z \leq 100 - 3x$ as x runs from 0 to 33. More precisely, the map

$$(x, y, z) \mapsto (y - x, z - x)$$

defines a bijection between the set

$$\{(x, y, z) \,|\, x, y, z \in \mathbb{Z}_{\geq 0}, x \leq y \leq z, x + y + z \leq 100\}$$

and the set

$$\bigcup_{0 \leq x \leq 33} \{(Y, Z) \,|\, Y, Z \in \mathbb{Z}_{\geq 0}, Y \leq Z, Y + Z \leq 100 - 3x\}.$$

By the above claim, the number of solutions of the given equation is

$$\begin{aligned}
&51^2 + 49 \cdot 50 + 48^2 + 46 \cdot 47 + 45^2 + 43 \cdot 44 + \cdots + 3^2 + 1 \cdot 2 \\
&= \sum_{k=0}^{16}(3k+1)(3k+2) + \sum_{k=1}^{17}(3k)^2 \\
&= \sum_{k=0}^{16}(9k^2 + 9k + 2) + \sum_{k=1}^{17} 9k^2 \\
&= 9 \times 17^2 + 2 \times 9 \times \frac{16 \cdot 17 \cdot 33}{6} + 9 \times \frac{16 \times 17}{2} + 2 \times 17 \\
&= 9 \times 17\,(17 + 176 + 8) + 34 \\
&= 153 \times 201 + 34 \\
&= 30600 + 187 \\
&= 30787.
\end{aligned}$$

■

§1.4 Considering cases

Example 1.7 (India RMO 1997 P3)**.** Solve for real x:

$$\frac{1}{[x]} + \frac{1}{[2x]} = x - [x] + \frac{1}{3},$$

where $[x]$ is the greatest integer less than or equal to x (eg, $[3.4] = 3$ and $[-0.8] = -1$).

Solution 8. Let x be a real number with $[x] \neq 0$ satisfying

$$\frac{1}{[x]} + \frac{1}{[2x]} = x - [x] + \frac{1}{3}.$$

Let n denote the integer $[x]$.

If $n \leq x < n + \frac{1}{2}$, then

$$\frac{1}{n} + \frac{1}{2n} = x - n + \frac{1}{3},$$

or equivalently, $x - n = \frac{3}{2n} - \frac{1}{3}$. Using the inequality $0 \leq x - n < \frac{1}{2}$, it follows that $\frac{9}{5} \leq n \leq \frac{9}{2}$, and hence x is equal to one of $2, 3, 4$. If $x = 2$, then

$$x = 2 + \frac{1}{2} + \frac{1}{4} - \frac{1}{3} = 2 + \frac{5}{12} = \frac{29}{12}.$$

If $x = 3$, then

$$x = 3 + \frac{1}{3} + \frac{1}{6} - \frac{1}{3} = \frac{19}{6}.$$

If $x = 4$, then

$$x = 4 + \frac{1}{4} + \frac{1}{8} - \frac{1}{3} = 4 + \frac{1}{24}.$$

If $n + \frac{1}{2} \leq x < n + 1$, then we obtain

$$\frac{1}{n} + \frac{1}{2n+1} = x - n + \frac{1}{3},$$

which gives

$$x - n = \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3}.$$

Using $x \geq n + \frac{1}{2}$, it follows that

$$\frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3} \geq \frac{1}{2},$$

or equivalently, $10n^2 - 7n - 6 < 0$, which yields $-\frac{1}{2} < n < \frac{6}{5}$. Since $n \neq 0$, we get $n = 1$, which implies that

$$x = n + \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3} = 2.$$

We conclude that x is equal to one of

$$\frac{29}{12}, \frac{19}{6}, \frac{97}{24}, 2.$$

Note that $\frac{29}{12}, \frac{19}{6}, \frac{97}{24}$ are solutions of the given equation. Also note that $x = 2$ does not satisfy the given equation. It follows that the required solutions are

$$\frac{29}{12}, \frac{19}{6}, \frac{97}{24}.$$

■

Example 1.8 (India RMO 2001 P3)**.** Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z, eg, $[\frac{7}{4}] = 1$).

Solution 9. Let x be a positive integer satisfying the given equation. Note that

$$\frac{x}{99} - \frac{x}{101} < 1$$

holds. Indeed, if a, b are real numbers satisfying $a - b \geq 1$, then

$$a \geq b + 1 \geq [b] + 1,$$

which yields $[a] \geq [b] + 1$. It follows that

$$x \leq \frac{99 \cdot 101}{2}.$$

Note that we have the following increasing sequence of integers

$$\begin{aligned} 0 &< 99 < 101 < 2 \cdot 99 < 2 \cdot 101 < 3 \cdot 99 < 3 \cdot 101 \\ &< \cdots \\ &< 49 \cdot 99 < 49 \cdot 101 < 50 \cdot 99 < 50 \cdot 101. \end{aligned}$$

since $n \cdot 101 < (n+1) \cdot 99$ holds for a positive integer n if $n < \frac{99}{2}$. Using $\frac{99 \cdot 101}{2} < 51 \cdot 99$, it follows that the positive integers satisfying the given equation lie strictly between 0 and $50 \cdot 101$.

Note that the given equation has no solution in the intervals

$$[99, 101), [2 \cdot 99, 2 \cdot 101), [3 \cdot 99, 3 \cdot 101), [4 \cdot 99, 4 \cdot 101), \dots, [49 \cdot 99, 49 \cdot 101), [50 \cdot 99, 50 \cdot 101).$$

Also note that any element of the intervals

$$[1, 99), [101, 2 \cdot 99), [2 \cdot 101, 3 \cdot 99), [3 \cdot 101, 4 \cdot 99), \cdots, [48 \cdot 101, 49 \cdot 99), [49 \cdot 101, 50 \cdot 99)$$

is a solution to the given equation.

Hence, the number of solutions of the given equation in positive integers is equal to

$$\begin{aligned}
&98 + (2 \cdot 99 - 101) + (3 \cdot 99 - 2 \cdot 101) + (4 \cdot 99 - 3 \cdot 101) \\
&\qquad + \cdots + (49 \cdot 99 - 48 \cdot 101) + (50 \cdot 99 - 49 \cdot 101) \\
&= 98 - 99 + 99(1 + 2 + 3 + 4 + \cdots + 49 + 50) - 101(1 + 2 + 3 + \cdots + 48 + 49) \\
&= -1 + 99 \cdot 25 \cdot 51 - 101 \cdot 49 \cdot 25 \\
&= -1 + 25(99 \cdot 51 - 99 \cdot 49 - 2 \cdot 49) \\
&= 2499.
\end{aligned}$$

■

Example 1.9 (India Pre-RMO 2012 P15)**.** How many nonnegative integral values of x satisfy the equation $\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]$?

Solution 10. If x is a nonnegative integer satisfying $\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]$, then

$$\frac{x}{5} - \frac{x}{7} < 1$$

holds. Indeed, if a, b are real numbers satisfying $a - b \geq 1$, then

$$a \geq b + 1 \geq [b] + 1,$$

which yields $[a] \geq [b] + 1$. It follows that

$$x \leq \frac{35}{2}.$$

Note that we have the following increasing sequence of integers

$$0 < 5 < 7 < 2 \cdot 5 < 2 \cdot 7 < 3 \cdot 5 < 3 \cdot 7.$$

Using $\frac{35}{2} < 3 \cdot 7$, it follows that the positive integers satisfying the given equation lie strictly between 0 and $3 \cdot 7$.

Observe that none of the integers lying in the intervals

$$[5, 7), [2 \cdot 5, 2 \cdot 7), [3 \cdot 5, 3 \cdot 7)$$

satisfies the given equation. Also note that any element lying in the intervals

$$[0, 5), [7, 2 \cdot 5), [2 \cdot 7, 3 \cdot 5)$$

is a solution to the given equation.

It follows that the number of nonnegative integral solutions to the given equation is

$$5 + (2 \cdot 5 - 7) + (3 \cdot 5 - 2 \cdot 7) = 5 + 3 + 1 = 9.$$

■

§1.5 Arrange in order

Example 1.10 (Tournament of Towns Junior 1988 P3 by L. Tumescu, Romania)**.** Find all real solutions for the system

$$\begin{aligned}(x_3+x_4+x_5)^5 &= 3x_1,\\ (x_4+x_5+x_1)^5 &= 3x_2,\\ (x_5+x_1+x_2)^5 &= 3x_3,\\ (x_1+x_2+x_3)^5 &= 3x_4,\\ (x_2+x_3+x_4)^5 &= 3x_5.\end{aligned}$$

Solution 11. Let x_1, x_2, x_3, x_4, x_5 be real numbers satisfying the above system of equations.

> **Claim** — Let a, b be real numbers. Then $a-b$ is nonnegative if and only if so is a^5-b^5.

Proof of the Claim. Let ζ denote a 5-th root of unity other than 1. Note that

$$a^5-b^5 = (a-b)(a-b\zeta)(a-b\zeta^4)(a-b\zeta^2)(a-b\zeta^3) = (a-b)|a-b\zeta|^2|a-b\zeta^2|^2$$

holds. If $a-b$ is nonnegative, it follows that so is a^5-b^5.

If $a<b$ holds, then $a-b\zeta, a-b\zeta^2$ are nonzero, and hence,

$$|a-b\zeta|^2|a-b\zeta^2|^2 > 0,$$

which implies that $a^5<b^5$. This shows that if a^5-b^5 is nonnegative, then so is $a-b$. This completes the proof. □

Assume that $x_1 \geq x_2$ holds. Using the first two equations, we get $x_3 \geq x_1 \geq x_2$. Using the second and the third equations, we obtain $x_3 \geq x_1 \geq x_2 \geq x_4$. Considering the third and the fourth equations, it follows that $x_5 \geq x_3 \geq x_1 \geq x_2 \geq x_4$. The last two equations yield

$$x_5 \geq x_3 \geq x_1 \geq x_2 \geq x_4 \geq x_1,$$

which implies that x_1, x_2, x_3, x_4, x_5 are equal, and hence, $x_1 = 0$ or $x_1 = \pm 3^{1/4}$ holds.

If $x_1 \leq x_2$ holds, then replacing the x_i's by their negatives in the given equations, and using $-x_1 \geq -x_2$ along with an argument similar to the above, it follows that x_1, x_2, x_3, x_4, x_5 are equal, and $x_1 = 0$ or $x_1 = \pm 3^{1/4}$ holds.

Note that $(0,0,0,0,0)$ and $(\pm 3^{1/4}, \pm 3^{1/4}, \pm 3^{1/4}, \pm 3^{1/4}, \pm 3^{1/4})$ (where the signs correspond) satisfy the given system of equations.

This proves that the solutions to the given system of equations are $(0,0,0,0,0)$ and $(\pm 3^{1/4}, \pm 3^{1/4}, \pm 3^{1/4}, \pm 3^{1/4}, \pm 3^{1/4})$. ■

Example 1.11 (India INMO 1996 P3). Find all possible real numbers a, b, c, d, e which satisfy the following set of equations:

$$3a = (b + c + d)^3 \tag{2}$$
$$3b = (c + d + e)^3 \tag{3}$$
$$3c = (d + e + a)^3 \tag{4}$$
$$3d = (e + a + b)^3 \tag{5}$$
$$3e = (a + b + c)^3. \tag{6}$$

Solution 12. Let a, b, c, d, e be real numbers satisfying the given equations. Note that subtracting Eq. (3) from Eq. (2) yields

$$\begin{aligned}
3(a - b) &= (b - e)((b + c + d)^2 + (b + c + d)(c + d + e) + (c + d + e)^2) \\
&= (b - e)(3(c + d)^2 + (c + d)(2b + b + e + 2e) + (b^2 + be + e^2)) \\
&= (b - e)(3(c + d)^2 + 3(c + d)(b + e) + (b^2 + be + e^2)) \\
&= 3(b - e)((c + d)^2 + (c + d)(b + e) + (b^2 + be + e^2)/3) \\
&= 3(b - e)((c + d + (b + e)/2)^2 - ((b + e)/2)^2 + (b^2 + be + e^2)/3) \\
&= 3(b - e)((c + d + (b + e)/2)^2 + \frac{1}{12}(b^2 - 2be + e^2)),
\end{aligned}$$

that is,

$$a - b = (b - e)\left(\left(c + d + \frac{b + e}{2}\right)^2 + \frac{1}{12}(b - e)^2\right). \tag{7}$$

Similarly, subtracting Eq. (4) from Eq. (3), Eq. (5) from Eq. (4), Eq. (6) from Eq. (5), Eq. (2) from Eq. (6) give

$$b - c = (c - a)\left(\left(d + e + \frac{c + a}{2}\right)^2 + \frac{1}{12}(c - a)^2\right), \tag{8}$$

$$c - d = (d - b)\left(\left(e + a + \frac{d + b}{2}\right)^2 + \frac{1}{12}(d - b)^2\right), \tag{9}$$

$$d - e = (e - c)\left(\left(a + b + \frac{e + c}{2}\right)^2 + \frac{1}{12}(e - c)^2\right), \tag{10}$$

$$e - a = (a - d)\left(\left(b + c + \frac{a + d}{2}\right)^2 + \frac{1}{12}(a - d)^2\right). \tag{11}$$

Let us consider the case that $a = b$. Then $b = e$ holds by Eq. (7), and hence, we get $a = b = e$. This gives $a = d$ by Eq. (11), and we obtain $a = b = d = e$. Then $e = c$ holds by Eq. (10). This yields $a = b = c = d = e$. Using Eq. (2), we get $3a = (3a)^3$, which implies that $a = 0, \pm\frac{1}{3}$. So the solution (a, b, c, d, e) is equal to one of

$$(0,0,0,0,0), \left(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right),$$

where the signs correspond.

Now, let us consider the case that $a < b$. Since the squares are nonnegative, using Eq. (7), we get $a < b < e$. Using $a < e$ and Eq. (11), we get $d < a < b < e$. Applying Eq. (10) yields $d < a < b < e < c$. Combining Eq. (9) and $d < c$, we obtain $b < d < a < b < e < c$, which is impossible.

Note that if $a > b$ holds, then reversing the above inequalities, we would again obtain a contradiction.

The above argument proves that $a = b$, and hence, any solution (a, b, c, d, e) is equal to one of

$$(0,0,0,0,0), \left(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right).$$

Also note that any of the above is also a solution to the given equations. This proves that the above three tuples are precisely all the solutions. ■

Example 1.12 (Baltic Way 2010 P1)**.** Find all quadruples of real numbers (a, b, c, d) satisfying the system of equations

$$\begin{aligned}(b+c+d)^{2010} &= 3a\\ (a+c+d)^{2010} &= 3b\\ (a+b+d)^{2010} &= 3c\\ (a+b+c)^{2010} &= 3d.\end{aligned}$$

Solution 13. Let a, b, c, d, e be real numbers satisfying the given system of equations. Note that a, b, c, d, e are nonnegative.

If $a \geq b$ holds, then from the first two equations, we get $b \geq a$. Similarly, $a \leq b$ implies that $b \leq a$. This shows that $a = b$. Similar arguments imply that $b = c, c = d, d = a$. It follows that a, b, c, d, e are equal, and $a = 0$ or $a = 3^{1/2009}$ holds.

Also note that the tuples

$$(0,0,0,0,0), (3^{1/2009}, 3^{1/2009}, 3^{1/2009}, 3^{1/2009}, 3^{1/2009})$$

satisfy the given system of equations.

This proves that the solutions are precisely the above tuples. ■

Example 1.13 (India RMO 2012d P6)**.** Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1,$$
$$\frac{1}{yz} = \frac{y}{x} + 1,$$
$$\frac{1}{zx} = \frac{z}{y} + 1.$$

Solution 14. Let x, y, z be positive real numbers satisfying the given equations. It follows that $z = x^2y + xyz, x = y^2z + xyz, y = z^2x + xyz$, which gives

$$x - y = z(y^2 - zx), y - z = x(z^2 - xy), z - x = y(x^2 - yz).$$

Note that if two of x, y, z are equal, then all of them are equal. Assume that x, y, z are pairwise distinct. Without loss of generality[1], we may assume x is the maximum of x, y, z. If $x > y > z$ holds, then we obtain $y^2 > zx, z^2 > xy, x^2 < yz$, which gives $y^3 > xyz > x^3$ and contradicts $x > y$. On the other hand, $x > z > y$ implies $y^2 > zx, z^2 < xy, x^2 < yz$, which implies $x^3 < xyz < y^3$ and thus contradicts $x > y$. This shows that x, y, z cannot be distinct, and hence they are equal. This yields $x = 2x^3$, which gives $x = y = z = \frac{1}{\sqrt{2}}$. Note that the tuple $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ does satisfy the given system of equations. Hence, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is the only solution of the given system. ■

§1.6 Using inequalities

Example 1.14 (Moscow MO 1955 Grade 9 P3)**.** Find all real solutions of

$$\begin{cases} x^3 + y^3 &= 1, \\ x^4 + y^4 &= 1. \end{cases}$$

Solution 15. Let x, y be real numbers satisfying the above equations. Using $x^4 + y^4 = 1$, it follows that $-1 \leq x \leq 1, -1 \leq y \leq 1$. If some of x, y is negative, then

$$1 = x^3 + y^3 < \max|x|, |y|^3 \leq 1$$

holds, which is impossible. This shows that x, y are nonnegative. Using

$$x^4 \leq x^3, y^4 \leq y^3, x^4 + y^4 = x^3 + y^3,$$

we obtain $x^4 = x^3$. It follows that $x = 0$ or $x = 1$, and hence (x, y) is equal to $(0, 1)$ or $(1, 0)$. Note that the pairs $(0, 1), (1, 0)$ also satisfy the given equations. Hence, the solutions to the given equations are $(0, 1), (1, 0)$. ■

[1] Is it clear that there is no loss of generality?

Example 1.15. Find the real solutions of the equation $3^x + 4^x = 5^x$.

Solution 16. Note that if $n \geq 3$, then $x^n \leq x^3 < x^2$ if $0 < x < 1$ (since $x^2 - x^3 = x^2(1-x) > 0, x^3 - x^n = x^3(1 - x^{n-3}) \geq 0$). In particular,

$$\left(\frac{3}{5}\right)^n < \left(\frac{3}{5}\right)^2, \left(\frac{4}{5}\right)^n < \left(\frac{4}{5}\right)^2.$$

Adding we get $(\frac{3}{5})^n + (\frac{4}{5})^n < (\frac{3}{5})^2 + (\frac{4}{5})^2 = 1$. So $3^n + 4^n$ is not equal to 5^n if $n \geq 3$. Clearly the equality $3^n + 4^n = 5^n$ holds for $n = 2$ and does not hold for $n = 1$. This proves the result. ■

Exercise 1.16. Let m, n be distinct positive integers. Find all solutions of the system

$$x^m + y^m = 1, \quad x^n + y^n = 1$$

in nonnegative reals.

Example 1.17 (Moscow MO 1957 Grade 8 P4)**.** Solve the system:

$$\frac{2x^2}{1+x^2} = y, \quad \frac{2y^2}{1+y^2} = z, \quad \frac{2z^2}{1+z^2} = x.$$

Solution 17. Let x, y, z be real numbers satisfying the given equations. Note that x, y, z are nonnegative. If one of them is equal to 0, it follows that all of them are equal to 0.

Let us assume that $xyz \neq 0$. Note that

$$x \geq \frac{y}{2}\left(x + \frac{1}{x}\right) \geq y,$$

and similarly, it follows that $y \geq z$ and $z \geq x$. This yields $x = y = z$, and hence $x = 1$.

Note that $(0,0,0), (1,1,1)$ also satisfy the given equations.

This shows that the solutions to the given system are precisely $(0,0,0)$ and $(1,1,1)$. ■

Example 1.18 (Asian Pacific MO 2003 P1)**.** Let a, b, c, d, e, f be real numbers such that the polynomial

$$x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

has eight positive real roots. Determine all possible values of f.

Solution 18. Let $x_1, \ldots, x_8$ denote the roots of this polynomial. Note that

$$\sum_{i=1}^{8} x_i = 4, \qquad \sum_{1\leq i<j\leq 8} x_i x_j = 7$$

holds, which imply

$$\sum_{i=1}^{8} x_i^2 = 4^2 - 2\cdot 7 = 2.$$

Applying the QM-AM inequality, we obtain

$$\sqrt{\frac{x_1^2 + \cdots + x_8^2}{8}} \geq \frac{x_1 + \cdots + x_8}{8}.$$

Note that this is an equality. This implies that the roots $x_1, \ldots, x_8$ are equal to $\frac{4}{8} = \frac{1}{2}$. This yields $f = 1/256$. ■

Example 1.19 (India RMO 2011a P6)**.** Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

Solution 19. By AM-GM inequality, we have

$$1 \geq 2\sqrt{16^{x^2+y}\cdot 16^{x+y^2}} = 2^{1+2(x^2+y+x+y^2)},$$

which implies that $x^2+y^2+x+y+\frac{1}{2} \leq 0$. This shows that $\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 \leq 0$. Since x, y are real numbers, we get $x = y = -\frac{1}{2}$.

Note that the pair $(-\frac{1}{2}, -\frac{1}{2})$ satisfies the given equation. Hence, $(-\frac{1}{2}, -\frac{1}{2})$ is the only solution to the given equation. ■

Example 1.20 (India RMO 2014c P4)**.** Find all positive reals x, y, z such that

$$\begin{aligned} 2x - 2y + \frac{1}{z} &= \frac{1}{2014}, \\ 2y - 2z + \frac{1}{x} &= \frac{1}{2014}, \\ 2z - 2x + \frac{1}{y} &= \frac{1}{2014}. \end{aligned}$$

Solution 20. Let x, y, z be positive real numbers satisfying the given equations. Adding these equations yields

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014}.$$

Multiplying these equations by z, x, y respectively and then by adding them, we obtain

$$3 = \frac{x}{2014} + \frac{y}{2014} + \frac{z}{2014}.$$

This implies that

$$\frac{x+y+z}{3} = \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

holds. Since x, y, z are positive real numbers, by AM-HM inequality, it follows that x, y, z are equal. Hence x, y, z are equal to 2014.

Note that $(x, y, z) = (2014, 2014, 2014)$ satisfies the given equations. It follows that $(2014, 2014, 2014)$ is the only solution of the given system of equations over the positive reals. ■

Example 1.21 (KöMaL B4745, proposed by L. Longáver, Szatmárnémeti)**.** Let n be a positive integer. Solve the equation

$$\frac{1}{\sin^{2n} x} + \frac{1}{\cos^{2n} x} = 2^{n+1}.$$

Solution 21. Let x be a real number such that $\sin x \cos x \neq 0$ and the given equation holds. Note that

$$\frac{1}{\sin^{2n} x} + \frac{1}{\cos^{2n} x} \geq \frac{2}{\sin^n x \cos^n x}$$

holds, which implies that

$$\begin{aligned} 1 &\leq 2^n \sin^n x \cos^n x \\ &= \sin^n 2x \\ &\leq 1, \end{aligned}$$

and hence, we obtain $\sin 2x = 1$. It follows that $x = \frac{1}{2}(k\pi \pm \frac{\pi}{2})$ for some integer k.

Also note that any real number of the form $\frac{1}{2}(k\pi \pm \frac{\pi}{2})$ satisfies the given equation. Hence, the solutions of the given equation are precisely the real numbers of the form $\frac{1}{2}(k\pi \pm \frac{\pi}{2})$. ■

Example 1.22. Find all real numbers x such that $\sin^3 x + \cos^4 x = 1$.

Solution 22. Let x be a real number satisfying $\sin^3 x + \cos^4 x = 1$. Note that

$$\begin{aligned} 1 &= \sin^3 x + \cos^4 x \\ &\leq |\sin^3 x| + \cos^4 x \\ &\leq \sin^2 x + \cos^2 x \end{aligned}$$

$$= 1$$

holds, which shows that

$$\sin^3 x = |\sin^3 x| = \sin^2 x, \cos^4 x = \cos^2 x.$$

It follows that

$$\sin x = 0, \cos x = \pm 1$$

holds, or

$$\sin x = 1$$

holds. This implies that x lies in

$$\{n\pi \,|\, n \in \mathbb{Z}\} \cup \{2n\pi \pm \frac{\pi}{2} \,|\, n \in \mathbb{Z}\}.$$

Also note that any element of the above set satisfies the given equation. It follows that the set of solutions is equal to tbe above set. ∎

Example 1.23. Show that $e^x + x^2 = \sin x$ admits no real solutions.

Solution 23. On the contrary, let us assume that some real number x satisfies the above equation. Note that if $x > 0$, then

$$\sin x = e^x + x^2 > 1,$$

which is impossible. This shows that $x \leq 0$. Note that $x \neq 0$. It follows that

$$x^2 \leq x^2 + e^x = \sin x \leq 1,$$

which gives $-1 \leq x < 0$, and this shows that $\sin x < 0$, and consequently, $0 < e^x + x^2 = \sin x < 0$ holds, which is impossible. Hence, the given equation admits no real solution. ∎

§1.7 Solving a linear combination

Example 1.24 (All-Russian MO 1991 Grade 10 P7, India RMO 1994 P4)**.** Solve the system of equations for real x and y:

$$5x\left(1 + \frac{1}{x^2+y^2}\right) = 12, \quad 5y\left(1 - \frac{1}{x^2+y^2}\right) = 4.$$

Solution 24. Let x, y be real numbers with $(x, y) \neq 0$, and satisfying the given equations. To make use of the factorization $x^2 + y^2 = (x+iy)(x-iy)$, we add the second equation multiplied by i to the first to obtain

$$5\left(x + iy + \frac{1}{x+iy}\right) = 12 + 4i.$$

Substituting $z = x + iy$, we obtain

$$5z^2 - (12 + 4i)z + 5 = 0,$$

which gives

$$\begin{aligned} z &= \frac{1}{10}(12 + 4i \pm \sqrt{(12+4i)^2 - 100}) \\ &= \frac{1}{10}(12 + 4i \pm 2\sqrt{7 + 24i}) \\ &= \frac{1}{10}(12 + 4i \pm 2(4 + 3i)). \end{aligned}$$

So

$$z = 2 + i, \frac{2}{5} - \frac{i}{5}.$$

Thus we get $(x, y) = (2, 1), (\frac{2}{5}, -\frac{1}{5})$.

■

Example 1.25 (India RMO 1996 P3)**.** Solve for real number x and y:

$$xy^2 = 15x^2 + 17xy + 15y^2, \quad x^2y = 20x^2 + 3y^2.$$

Solution 25. Adding the first equation multiplied by i to the second, it follows that the given system of equations over the reals is equivalent to

$$x^2y + ixy^2 = (20 + 15i)x^2 + 17ixy + (3 + 15i)y^2,$$

which is equivalent to

$$xy(x + iy) = (x + iy)((20 + 15i)x - i(3 + 15i)y),$$

which simplies to

$$(x + iy)(20x + 15y - xy + 3i(5x - y)) = 0.$$

Note that any solution of the above equal is equal to one of $(0, 0), (19, 95)$, and conversely, any of $(0, 0), (19, 95)$ satisfies the above equation. Hence, the solutions of the given system of equations over the reals are precisely $(0, 0), (19, 95)$. ■

§1.8 Interchanging unknowns and parameters

We refer to [**AE11**, §1.4] for further problems.

Example 1.26 (India RMO 2000 P7)**.** Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

Solution 26. Note that

$$\begin{aligned}
&x^4 - 2ax^2 + x + a^2 - a \\
&= a^2 - (1 + 2x^2)a + x + x^4 \\
&= (a - x^2 + x - 1)(a - x^2 - x) \\
&= (x^2 - x + 1 - a)(x^2 + x - a) \\
&= \left(\left(x - \frac{1}{2}\right)^2 - \left(a - \frac{3}{4}\right)\right)\left(\left(x + \frac{1}{2}\right)^2 - \left(a + \frac{1}{4}\right)\right),
\end{aligned}$$

which has four real roots if and only if $a - \frac{3}{4}, a + \frac{1}{4}$ are both nonnegative, that is, when $a \geq \frac{3}{4}$. ■

§1.9 $x + 1/x$

We refer to [**AG09**, §2.8] for further problems.

Example 1.27 (cf. Putnam 1995 B4)**.** Let x denote the root of the equation $L + L^{-1} = 2207$ such that $x > 1$. Find $x^{1/8}$.

Solution 27. Note that if b is a real number satisfying $b \geq 2$, and α is the larger root of $x^2 - bx + 1$, then $\alpha^{1/2}$ is the larger root of $x^2 - \sqrt{b+2}x + 1$. Also note that $\sqrt{b+2} \geq 2$ holds.

Since x is the larger root of $y^2 - 2207y + 1$ and $2207 \geq 2$, it follows that $x^{1/2}$ is the larger root of $y^2 - 47x + 1$. Since $47 \geq 2$, we conclude that $x^{1/4}$ is the larger root of $y^2 - 7x + 1$. It follows that $x^{1/8}$ is the larger root of $y^2 - 3y + 1$, that is, $x^{1/8}$ is equal to $\frac{3+\sqrt{5}}{2}$. ■

§1.10 Newton's identities

Example 1.28. Let n be a positive integer. Prove that the only complex solution to the following system of equations:

$$x_1 + x_2 + \cdots + x_n = n, x_1^2 + x_2^2 + \cdots + x_n^2 = n, \cdots, x_1^n + x_2^n + \cdots + x_n^n = n$$

is $x_1 = x_2 = \cdots = x_n = 1$.

Solution 28. By the given condition, the power sums associated with $x_1, \ldots, x_n$ coincides with the power sums associated with $1, \ldots, 1$. By Newton's identities, elementary symmetric functions can be expressed in terms of power sums. So elementary symmetric functions associated with $x_1, \ldots, x_n$ coincides with the elementary symmetric functions associated with $1, \ldots, 1$. Hence, the polynomial $(x - x_1)(x - x_2) \ldots (x - x_n)$ is equal to $(x - 1)^n$, which shows that $x_1, \ldots, x_n$ are equal to 1. ■

§1.11 Intermediate value theorem

Example 1.29 (India BStat-BMath 2015)**.** If $0 < a_1 < \cdots < a_n$, show that the following equation has exactly n roots.

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = 2015.$$

Walkthrough —

(a) Show that for any $1 \leq i \leq n$,

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} - 2015$$

changes signs around the "punctured interval" $(a_i - \varepsilon_i, a_i + \varepsilon_i) \setminus \{a_i\}$.

(b) Show that it has a root in (a_i, a_{i+1}) for any $1 \leq i < n$.

Solution 29. Note that if $n = 1$, then we are done. Henceforth, let us assume that $n \geq 2$ holds. Let us establish the following Claim.

Claim — For any $1 \leq i \leq n$, there exists $\varepsilon_i > 0$ such that

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} - 2015$$

is well-defined on $(a_i - \varepsilon_i, a_i + \varepsilon_i) \setminus \{a_i\}$, and takes positive values on $(a_i - \varepsilon_i, a_i)$ and negative values on $(a_i, a_i + \varepsilon_i)$.

Proof of the Claim. Let $1 \leq i \leq n$ be an integer. If $1 \leq j \leq n$ is an integer and $j \neq i$, then for any real number x satisfying $|a_i - x| \leq \frac{1}{2}|a_j - a_i|$, note that

$$\begin{aligned} |a_j - x| &= |a_j - a_i| - |a_i - x| \\ &\geq \frac{1}{2}|a_j - a_i| \end{aligned}$$

holds, which yields

$$\begin{aligned} \frac{a_j}{a_j - x} &\geq -\frac{|a_j|}{|a_j - x|} \\ &\geq -\frac{2|a_j|}{|a_j - a_i|}, \end{aligned}$$

and

$$\frac{a_j}{a_j - x} \leq \frac{|a_j|}{|a_j - x|}$$

$$\leq \frac{2|a_j|}{|a_j - a_i|}.$$

Since $a_1, \dots, a_n$ are pairwise distinct, it follows that

$$\varepsilon_i := \min \left\{ \frac{1}{2} \min_{1 \leq j \leq n, j \neq i} \{|a_j - a_i|\}, \frac{a_i}{2015 + \sum_{1 \leq j \leq n, j \neq i} \frac{2|a_j|}{|a_j - a_i|}} \right\}$$

is positive. For any x lying in $(a_i - \varepsilon_i, a_i)$, it follows that

$$\begin{aligned} & \frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \dots + \frac{a_n}{a_n - x} - 2015 \\ & > \frac{a_i}{\varepsilon_i} + \sum_{j \neq i} \frac{a_j}{a_j - x} - 2015 \quad (\text{using } a_j > 0) \\ & \geq \frac{a_i}{\varepsilon_i} - \sum_{j \neq i} \frac{2|a_j|}{|a_j - a_i|} - 2015 \\ & \geq 0, \end{aligned}$$

and for any x lying in $(a_i, a_i + \varepsilon_i)$, we obtain

$$\begin{aligned} & \frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \dots + \frac{a_n}{a_n - x} - 2015 \\ & < -\frac{a_i}{\varepsilon_i} + \sum_{j \neq i} \frac{a_j}{a_j - x} - 2015 \quad (\text{using } a_j > 0) \\ & \leq -\frac{a_i}{\varepsilon_i} + \sum_{j \neq i} \frac{2|a_j|}{|a_j - a_i|} + 2015 \\ & \leq 0. \end{aligned}$$

This proves the Claim. □

Note that $a_1 < a_2 < \dots < a_n$, and applying the above claim, it follows that for any $1 \leq i < n$,

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \dots + \frac{a_n}{a_n - x} - 2015$$

defines a well-defined function on (a_i, a_{i+1}), which takes negative and positive values. By the intermediate value theorem, this functions vanishes at a point lying in (a_i, a_{i+1}).

Note that

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \dots + \frac{a_n}{a_n - x} - 2015$$

can be simplified to a polynomial equation of degree n with real coefficients, and by the above argument, it has a root in (a_i, a_{i+1}) for any $1 \leq i < n$, and thus it has at least $n - 1$ real roots, and hence it has precisely n real roots. This completes the proof. ■

§1.12 Miscellaneous

Example 1.30 (India Pre-RMO 2012 P9)**.** Suppose that $4^{X_1} = 5, 5^{X_2} = 6, 6^{X_3} = 7, \cdots, 126^{X_{123}} = 127, 127^{X_{124}} = 128$. What is the value of the product $X_1 X_2 \cdots X_{124}$?

Solution 30. It follows that

$$4^{X_1 X_2 \cdots X_{124}} = 128,$$

which gives $X_1 X_2 \cdots X_{124} = 7/2$. ■

Example 1.31 (India RMO 2012f P3)**.** Solve

$$2^{5x+2[x]} = 11 \cdot 2^{5x} + 5 \cdot 2^{7[x]}$$

for $x \in \mathbb{R}, \mathbb{Q}$.

> **Walkthrough** — Show that
>
> $$2^{5\{x\}} = \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}$$
>
> holds, where $\{x\}$ denotes the fractional part of x. For any real x satisfying the above, prove that $[x] \geq 2$ holds and the RHS of the above lies between 1 and 2^5.

Solution 31. Let x be a real number satisfying the above equation. Denote by $\{x\}$ the fractional part of x, that is $\{x\}$ is equal to $x - [x]$. Replacing x by $[x] + \{x\}$ in the given equation yields

$$2^{7[x]+5\{x\}} = 11 \cdot 2^{5[x]+5\{x\}} + 5 \cdot 2^{7[x]},$$

which implies that $2^{2[x]+5\{x\}} = 11 \cdot 2^{5\{x\}} + 5 \cdot 2^{2[x]}$, which gives

$$2^{5\{x\}} = \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}.$$

Since $2^{5\{x\}}$ is positive, it follows that $2^{2[x]} - 11$ is positive, and hence $[x] \geq 2$. Note that for any integer $n \geq 2$,

$$5 < \frac{5 \cdot 2^{2n}}{2^{2n} - 11} = 5\left(1 + \frac{11}{2^{2n} - 11}\right) \leq 5\left(1 + \frac{11}{5}\right) = 16 < 2^5,$$

holds, and hence $\frac{5 \cdot 2^{2n}}{2^{2n} - 11}$ is equal to 2^{5y} for some $0 < y < 1$. It follows that

$$x = [x] + \log_{32} \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}, \text{ with } 0 < \log_{32} \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11} < 1.$$

It also follows that for any integer $n \geq 2$,

$$0 < \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n} - 11} < 1$$

holds, and hence $n + \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ satisfies the given equation. Consequently, the real solutions are of the form $n + \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ for some integer $n \geq 2$.

Note that if $n + \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ is rational for some integer $n \geq 2$, then $r := \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ is rational and $2^{5r} = \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ is rational. By the fundamental theorem of arithmetic, it follows that $5r = 2n$. Since r lies between 0 and 1, we obtain $r = 4/5$. Substituting $r = 4/5$ in $2^{5r} = \frac{5 \cdot 2^{2n}}{2^{2n}-11}$, we get $n = 2$. So, any rational solution to the given equation is equal to $2 + 4/5$. Also note that $2 + 4/5$ satisfies the given equation. Thus the only rational solution is $2 + \frac{4}{5} = \frac{14}{5}$. ■

Example 1.32 (India RMO 2015a P6)**.** Show that there are infinitely many positive real numbers a, which are not integers, such that $a(a - 3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Solution 32. If n is a positive integer and $0 \leq x < 1$, then $(n + x)(n + x - 3\{n + x\})$ is an integer if and only if $(n + x)(n - 2x) = n^2 - nx - 2x^2$ is an integer. Note that if n is odd and $x = \frac{1}{2}$, then $n^2 - nx - 2x^2$ is an integer, and hence, so is $(n + x)(n + x - 3\{n + x\})$. It follows that for any rational number a of the form $2k + 1 + \frac{1}{2}$, where k is a nonnegative integer, the real number $a(a - 3\{a\})$ is also an integer. This completes the proof. ■

Example 1.33 (India RMO 2015d P6)**.** Find all real numbers a such that $4 < a < 5$ and $a(a - 3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Solution 33. Let a be a real number such that $4 < a < 5$ and $a(a - 3\{a\})$ is an integer. Denote by x the fractional part of a, that is, $x = a - 4$. So $a(a - 3\{a\})$ is equal to $(4 + x)(4 - 2x) = 16 - 4x - 2x^2$, which is an integer. Thus $2x^2 + 4x = k$ for some integer k. Since $0 < x < 1$, it follows that $0 < k < 6$. Note that x is equal to

$$\frac{1}{4}(-4 \pm \sqrt{16 + 8k}) = -1 \pm \sqrt{1 + k/2}.$$

Since x is positive, it is equal to $-1 + \sqrt{1 + k/2}$. It follows that a is equal to $3 + \sqrt{1 + k/2}$, for some integer $0 < k < 6$. Also note that any real number of this form satisfies the required condition. Hence, the real numbers lying in $(4, 5)$ are precisely

$$3 + \sqrt{3/2}, 3 + \sqrt{2}, 3 + \sqrt{5/2}, 5, 3 + \sqrt{7/2}.$$

■

Example 1.34 (India RMO 2015e P6)**.** Find all real numbers a such that $3 < a < 4$ and $a(a - 3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Example 1.35 (Formula of Unity 2013/2014 Round 2 R10 P2)**.** Let $f(x) = x^3 + 9x^2 + 27x + 24$. Solve the equation $f(f(f(f(x)))) = 0$.

Solution 34. Note that

$$f(x) = (x + 3)^3 - 3,$$

this gives

$$f(f(x)) = (x + 3)^9 - 3,$$

and this yields

$$f(f(f(f(x)))) = (f(f(x)) + 3)^9 - 3 = (x + 3)^{81} - 3.$$

Hence, the solutions of $f(f(f(f(x)))) = 0$ are

$$3^{1/81}\zeta^i - 3, 1 \leq i \leq 81,$$

where ζ denotes $\cos\frac{2\pi}{81} + i\sin\frac{2\pi}{81}$. ■

Example 1.36 (Formula of Unity 2015/2016 Round 1 R9 R10 R11 P7)**.** It is well known that $3^2 + 4^2 = 5^2$. It is less known that $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Is it true that for any positive integer k there are $2k + 1$ consecutive positive integers such that the sum of the squares of the first $k + 1$ of them equals the sum of the squares of the rest k?

Solution 35.

■

Example 1.37 (All-Russian MO 1995 Grade 10 P1 by V. Senderov, I. Yashchenko)**.** Solve the equation $\cos\cos\cos\cos x = \sin\sin\sin\sin x$.

Solution 36. Let us establish the following claim.

Claim — For any real number x, the inequality

$$\cos\cos x > |\sin\sin x|$$

holds.

Proof of the Claim. Note that it suffices to prove the inequality for the positive reals. Also note that it suffices to prove it for x lying in $(0, 2\pi)$. Moreover, for any $x \in \mathbb{R}$,

$$\begin{aligned}\cos\cos(x+\pi) &= \cos\cos x,\\ |\sin\sin(x+\pi)| &= |\sin(-\sin x)|\\ &= |\sin\sin x|\end{aligned}$$

holds. Hence, it is enough to prove the inequality for x lying in $(0, \pi)$. Note that for any $\theta \in \mathbb{R}$,

$$\begin{aligned}\cos\cos\left(\frac{\pi}{2}+\theta\right) &= \cos\cos\left(\frac{\pi}{2}-\theta\right),\\ \sin\sin\left(\frac{\pi}{2}+\theta\right) &= \sin\sin\left(\frac{\pi}{2}-\theta\right)\end{aligned}$$

holds. This reduces to proving the inequality for x lying in $(0, \pi/2)$. In the following, x denotes a real number lying between 0 and $\pi/2$. Note that

$$\begin{aligned}\cos\cos x &\geq 1 - \frac{\cos^2 x}{2},\\ |\sin\sin x| &= \sin\sin x\\ &\leq \sin x - \frac{\sin^3 x}{6} + \frac{\sin^5 x}{120}\end{aligned}$$

holds. Observe that

$$\begin{aligned}&1 - \frac{\cos^2 x}{2} - \left(\sin x - \frac{\sin^3 x}{6} + \frac{\sin^5 x}{120}\right)\\ &= \frac{1+\sin^2 x}{2} - \left(\sin x - \frac{\sin^3 x}{6} + \frac{\sin^5 x}{120}\right)\\ &= \frac{(1-\sin x)^2}{2} + \frac{\sin^3 x}{6}\left(1 - \frac{\sin^2 x}{20}\right)\\ &> 0.\end{aligned}$$

This proves that

$$\cos\cos x > |\sin\sin x|.$$

□

Using the above Claim, it follows that for any real number x,

$$\begin{aligned}|\sin\sin x| &\leq \cos\cos x\\ &\leq \cos 1\end{aligned}$$

holds, which implies that

$$\cos\cos\cos\cos x$$

$$
\begin{aligned}
&> \sin \sin \cos \cos x \quad \text{(using the Claim)} \\
&> \sin \sin |\sin \sin x| \quad \text{(since } \sin \sin \text{ is increasing on } [0, \cos 1]) \\
&= |\sin \sin \sin \sin x| \\
&\geq \sin \sin \sin \sin x.
\end{aligned}
$$

■

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