Roots of unity

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Roots of unity

Some of the problems have been borrowed from Yufei Zhao's handout on polynomials.

Example 1.1 (India Pre-RMO 2012 P17)**.** Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

See also Example 1.2, USAMO 2014 P1.

Solution 1. Let $P(x)$ denote the polynomial $x^3 + 3x + 5$. Note that

$$\begin{aligned}
&\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right) \\
&= \frac{1}{x_1x_2x_3}(x_1^2+1)(x_2^2+1)(x_3^2+1) \\
&= \frac{1}{x_1x_2x_3}(x_1+i)(x_2+i)(x_3+i)(x_1-i)(x_2-i)(x_3-i) \\
&= \frac{1}{x_1x_2x_3}P(-i)P(i) \\
&= \frac{1}{-5}\left|P(i)\right|^2 \\
&= \frac{1}{-5}\left|5-2i\right|^2 \\
&= -\frac{29}{5}.
\end{aligned}$$

■

Example 1.2 (USAMO 2014 P1)**.** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product

$$(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$$

can take.

See also Example 1.1, India Pre-RMO 2012 P17.

Solution 2. Note that

$$\begin{aligned}(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1) &= P(i)P(-i)\\ &= (1-b+d)^2+(a-c)^2\\ &= (b-d-1)^2+(a-c)^2\\ &\geq 16.\end{aligned}$$

Taking $a = c$ and $b = 5, d = 0$, we obtain

$$(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1) = 16.$$

Hence, the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take is equal to 16. ■

Example 1.3 (USAMO 1976 P5)**.** If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4+x^3+x^2+x+1)S(x),$$

prove that $x-1$ is a factor of $P(x)$.

Solution 3. Denote the 5-th root of unity $\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ by ζ. Substituting $x = \zeta, \zeta^2, \zeta^3$, we obtain

$$\begin{aligned}P(1)+\zeta Q(1)+\zeta^2R(1) &= 0,\\ P(1)+\zeta^2Q(1)+\zeta^4R(1) &= 0,\\ P(1)+\zeta^3Q(1)+\zeta^6R(1) &= 0.\end{aligned}$$

Eliminating $R(1)$ from the first two equations yields

$$(1-\zeta^2)P(1)+\zeta^2(1-\zeta)Q(1) = 0,$$

and eliminating $R(1)$ from the last two equations yields

$$(1-\zeta^2)P(1)+\zeta^3(1-\zeta)Q(1) = 0.$$

Eliminating $Q(1)$ from the above two equations, we obtain $(1-\zeta)P(1) = 0$, which gives $P(1) = 0$. This shows that $x-1$ is a factor of $P(x)$. ■

Example 1.4 (Leningrad Math Olympiad 1991)**.** A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form

$$a_k + a_{k+p} + a_{k+2p} + \ldots$$

is the same for any $k = 1, 2, 3, \ldots$. For instance the sequence

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 3, a_6 = 2$$

is 3-balanced. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

Summary — Consider the polynomial $\sum_{i=1}^{n} a_i x^n$.

Solution 4. Let $a_1, a_2, \ldots, a_{50}$ be a sequence. Assume that it is p-balanced for $p \in \{3, 5, 7, 11, 13, 17\}$. For an integer $n \geq 1$, denote the root of unity $\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ by ζ_n. Let $P(x)$ denote the polynomial $\sum_{i=1}^{n} a_i x^i$.

Let $3 \leq p \leq 17$ be a prime. Since $a_1, a_2, \ldots, a_{50}$ is p-balanced, for any $1 \leq \ell < p$, we obtain

$$\begin{aligned} P(\zeta_p^\ell) &= \sum_{k=1}^{p} (a_k + a_{k+p} + \ldots) \zeta_p^{k\ell} \\ &= (a_1 + a_{1+p} + \ldots) \sum_{k=1}^{p} \zeta_p^{k\ell} \\ &= 0, \end{aligned}$$

where the final equality follows since $\zeta_p^\ell \neq 1$. This shows that the polynomial $P(x)$ vanishes at the elements of the set

$$\cup_{p \in \{3,5,7,11,13,17\}} \{\zeta_p^\ell \,|\, 1 \leq \ell < p\},$$

which contains

$$\sum_{p \in \{3,5,7,11,13,17\}} (p-1) = 2 + 4 + 6 + 10 + 12 + 16 = 50.$$

Moreover, $P(x)$ also vanishes at 0. Note that $P(x)$ is a polynomial of degree 50, and it has at least 51 zeroes. This gives that $P(x) = 0$, and hence, the terms of the sequence $a_1, a_2, \ldots, a_{50}$ are all equal to zero. ■

Example 1.5. Show that any function $f : \mathbb{R} \to \mathbb{R}$ can be written as a sum of an even function and an odd function.

Solution 5. Consider the functions $f_1, f_{-1} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_\varepsilon(x) = \frac{1}{2}(f(x) + \varepsilon f(\varepsilon x)), \qquad \text{for all } x \in \mathbb{R},$$

where $\varepsilon \in \{1, -1\}$. Note that f_1 is an even function, f_{-1} is an odd function, and that $f = f_1 + f_{-1}$ holds. ■

Remark. It seems from the above solution that we "knew" the functions f_1, f_{-1} beforehand! Here is another solution to the above problem.

Let us assume that f can be expressed as the sum of an even function g and an odd function h. Note that

$$\begin{aligned} f(x) &= g(x) + h(x), \\ f(-x) &= g(-x) + h(-x) \end{aligned}$$

$$= g(x) - h(x)$$

hold for all $x \in \mathbb{R}$. Combining the above, we obtain

$$g(x) = \frac{1}{2}(f(x) + f(-x)),$$
$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

for all $x \in \mathbb{R}$.

Note that this is **NOT a proof**. We have observed that

$$g(x) = \frac{1}{2}(f(x) + f(-x)),$$
$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

holds **relying on the assumption that** g is an even function, and h is an odd function satisfying $f = g + h$. This observation is **made under the hypothesis** that $f = g + h$ where g (resp. h) is even (resp. odd), and **cannot be a proof of the hypothesis**! However, this observation can be used to arrive at the above proof, by defining the functions f_1, f_{-1} as above, and showing that they have the required properties.

Example 1.6. Let ω denote the root of unity $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Show that any function $f : \mathbb{C} \to \mathbb{C}$ can be written as a sum of three functions $f_0, f_1, f_2 : \mathbb{C} \to \mathbb{C}$ satisfying

$$f_i(\omega z) = \overline{\omega}^i z \quad \text{for all } z \in \mathbb{C}.$$

Walkthrough — Assume that the function f can be expressed as stated, and then determine some of the properties of the functions f_0, f_1, f_2, as done in the above remark. What remains left to do?

Example 1.7. Let $P(x)$ be a monic polynomial with integer coefficients such that all its zeroes lie on the unit circle. Show that all the zeroes of $P(x)$ are roots of unity, that is, $P(x)$ divides $(x^n - 1)^k$ for some positive integers n, k.

Walkthrough —

(a) Use the fundamental theorem of symmetric polynomials, to prove the following claim.

> **Claim** — Let $f(x)$ be a monic polynomial of degree n with integer coefficients. Let $\alpha_1, \dots, \alpha_n$ denote its roots, counting multiplicities. Then for any integer $k \geq 1$, there is a monic polynomial of degree n with integer coefficients, having $\alpha_1^k, \alpha_2^k, \dots, \alpha_n^k$ as its roots.

(b) Applying the Claim, for each integer $k \geq 1$, obtain a monic polynomial

$P_k(x)$ with integer coefficients, having degree same as that of $P(x)$, whose roots, counted with multiplicities, are the k-th powers of the roots of $P(x)$.

(c) Note that the polynomials $P_1(x), P_2(x), \ldots$ are of the same degree, and the absolute values of the coefficients of any of them are bounded from the above by suitable binomial coefficients, which is smaller than 2^n, where n denotes the degree of $P(x)$. Since these polynomials have integer coefficients, by the pigeonhole principle, it follows that there is a positive integer k_0 such that $P_k(x) = P_{k_0}(x)$ holds for infinitely many positive integers k.

(d) Enumerate the roots of $P(x)$, and matching the roots of $P_k(x)$ with those of $P_{k_0}(x)$ (for suitable k's), we obtain a permutation of n letters. By the pigeonhole principle, infinitely many k's yield the same permutation, which implies that there are positive integers $k \neq \ell$ such that for any root of $P(x)$, its k-th and the ℓ-th powers are equal.