# Size of the roots

MOPSS

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Mathematics Olympiad

**Problem Solving Sessions**

MOPSS

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

## Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

## List of problems and examples



## §1 Size of the roots

Some of the following problems have been borrowed from [**Goy21**].

**Example 1.1.** Let $f(x)$ and $g(x)$ be nonconstant polynomials with real coefficients such that $f(x^2 + x + 1) = f(x)g(x)$. Show that $f(x)$ has even degree.

**Solution 1.** For any real root $\alpha$ of $f(x)$, note that $\alpha^2 + \alpha + 1$ is also a real root of $f(x)$ and $\alpha^2 + \alpha + 1 > \alpha$. Hence, if $f(x)$ possesses a real root, it would have real roots of arbitrarily large absolute value. Since $f(x)$ is nonconstant, it follows that $f(x)$ has no real roots. Since $f(x)$ is a polynomial with real coefficients, it follows that $f(x)$ is of even degree. ■

**Example 1.2.** Find all polynomials $P$ (with complex coefficients) satisfying

$$P(x)P(x+2) = P(x^2).$$

> **Summary** — Note that if $\alpha$ is a root of $P$, then so are $\alpha^2$ and $(\alpha-2)^2$. Considering absolute values, show that $P$ cannot have a root other than 1. Conclude that $P(x) = c(x-1)^n$.

**Solution 2.** Let $P$ be a polynomial with complex coefficients satisfying the given condition.

> **Claim** — If $P$ has a root $\alpha \neq 1$, then $P$ has a root which has absolute larger than the absolute value of $\alpha$.

*Proof of the Claim.* Note that if $|\alpha| \leq 1$, then

$$\begin{aligned} |(\alpha-2)^2| &\geq (2-|\alpha|)^2 \\ &\geq 1 \\ &\geq |\alpha|. \end{aligned}$$

Moreover, if $|(\alpha-2)^2| = |\alpha|$ holds, then $|\alpha| = 1$ and $\alpha = 2r$ for some real number $r \geq 0$, and hence, $\alpha = 1$. This shows that if $\alpha \neq 1$, and $|\alpha| \leq 1$, then $|(\alpha-2)^2| > |\alpha|$ holds. Also note that if $|\alpha| > 1$, then $|\alpha^2| > |\alpha|$ holds. Since $\alpha^2, (\alpha-2)^2$ are also roots of $P$, the Claim follows. □

If $P$ has a root other than 1, then applying the above Claim to a root of $P$ having the largest absolute value yields a contradiction. Hence, $P$ is a constant polynomial, or the only root of $P$ is equal to 1.

If $P$ is a constant, then $P$ is equal to 0 or 1. If $P$ is a nonconstant polynomial, then $P$ is equal to $c(x-1)^n$ for some $c \in \mathbb{C} \setminus \{0\}$, and an integer $n \geq 1$. Using the hypothesis, it follows that $c = 1$.

Observing that the polynomials $0, 1, (x-1)^n$ satisfy the given condition, we conclude that there are all the required polynomials. ■

**Example 1.3.** Does there exist a polynomial $f(x)$ satisfying

$$xf(x-1) = (x+1)f(x)?$$

Problem 4.12 of Putnam training problems by Miguel A. Lerma.

**Solution 3.** Note that the zero polynomial is the only constant polynomial which satisfies the given condition.

Suppose there exists a nonconstant polynomial $f(x)$ satisfying the given condition.

**Claim** — Any root of $f(x)$ is an integer.

*Proof of the Claim.* On the contrary, let us assume that $\alpha$ is root of $f(x)$, and that $\alpha$ is not an integer. Note that for any root of $\beta$ of $f(x)$ in $\mathbb{C} \setminus \mathbb{Z}$, the element $\beta - 1$ is also a root of $f(x)$ lying in $\mathbb{C} \setminus \mathbb{Z}$. It follows that for any integer $k \geq 1$, the element $\alpha - k$ is also a root of $f(x)$. Taking $k$ to be an integer larger than $|\alpha| + \sum_{\gamma \in \mathbb{C}, f(\gamma)=0} |\gamma|$, we obtain

$$|\alpha - k| \geq k - |\alpha| > |\gamma|,$$

and hence $f(x)$ has a root having absolute value larger than that of any of its roots, which is impossible. □

A similar argument can be used to prove that $f(x)$ does not vanish at any negative integer. Moreover, noting that if $f(x)$ vanishes at an integer $n \geq -1$, then $f(x)$ also vanishes at $n+1 \geq -1$, it can be proved using a similar argument that the roots of $f(x)$ are $\leq -2$. This contradicts the assumption that there exists a nonconstant polynomial satisfying the given condition. Consequently, the zero polynomial is the only polynomial that satisfies the given condition. ■

**Example 1.4.** Find all polynomials $P(x)$ satisfying

$$xP(x-1) = (x-20)P(x).$$

Problem 2 of the Problem session for October 28, Fall 2020, Putnam Club.

**Solution 4.** Note that the zero polynomial is the only constant polynomial which satisfies the given condition.

Suppose $P(x)$ is a nonconstant polynomial satisfying the given condition.

> **Claim —** Any root of $P(x)$ is an integer.

*Proof of the Claim.* On the contrary, let us assume that $\alpha$ is root of $P(x)$, and that $\alpha$ is not an integer. Note that for any root of $\beta$ of $P(x)$ in $\mathbb{C} \setminus \mathbb{Z}$, the element $\beta - 1$ is also a root of $P(x)$ lying in $\mathbb{C} \setminus \mathbb{Z}$. It follows that for any integer $k \geq 1$, the element $\alpha - k$ is also a root of $P(x)$. Taking $k$ to be an integer larger than $|\alpha| + \sum_{\gamma \in \mathbb{C}, P(\gamma)=0} |\gamma|$, we obtain

$$|\alpha - k| \geq k - |\alpha| > |\gamma|,$$

and hence $P(x)$ has a root having absolute value larger than that of any of its roots, which is impossible. □

A similar argument can be used to prove that no root of $P(x)$ is $\leq -1$ or $\geq 20$. In other words, the roots of $P(x)$ are among $0, 1, 2, \ldots, 19$. The given condition implies that if $0 \leq i \leq 19$ is a root of $P(x)$, then all of the integers $0, 1, 2, \ldots, 19$ are roots of $P(x)$. Hence, $P(x)$ is equal to $cx^{n_0}(x-1)^{n_1} \ldots (x-19)^{n_{19}}$ for some $c \in \mathbb{C}$, and positive integers $n_0, n_1, \ldots, n_{19}$. Using the condition $xP(x-1) = (x-20)P(x)$, it follows that the integers $n_0, \ldots, n_{19}$ are equal to 1. This shows that $P(x)$ is equal to $cx(x-1)(x-2)\ldots(x-20)$. Moreover, any polynomial of this form satisfies the given condition. Hence, the polynomials satisfying the given condition are precisely of the above form. ■

**Example 1.5** (India INMO 2018 P4)**.** Find all polynomials $P(x)$ with real coefficients such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

> **Walkthrough —**
>
> **(a)** Show that if $\alpha$ is a root of $P(x)$, then $P(x)$ vanishes at $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$, where $\beta_1, \beta_2$ are the roots of $x^2 + x + 1 = \alpha$.
>
> **(b)** If $\alpha$ is nonzero, then show that one of $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$ is larger than $\alpha$ in absolute value.

**Solution 5.** Let us establish the following claim.

> **Claim —** Let $\alpha$ denote a nonzero root of $P(x)$ in $\mathbb{C}$. Then for some $\beta \in \mathbb{C}$ satisfying $\beta^2 + \beta + 1 = \alpha$, the element $(\beta - 1)\alpha$ is a root of $P(x)$

and is larger than $\alpha$ in absolute value.

*Proof of the Claim.* Let $\beta_1, \beta_2$ denote the roots of $x^2 + x + 1 = \alpha$ in $\mathbb{C}$. Since $P(x^2+x+1)$ divides the polynomial $P(x^3-1)$, it follows that for any $1 \leq i \leq 2$,

$$\beta_i^3 - 1 = (\beta_i - 1)(\beta_i^2 + \beta_i + 1) = (\beta_i - 1)\alpha$$

is a root of $P(x)$. Noting that

$$\begin{aligned}|\beta_1 - 1| + |\beta_2 - 1| &\geq |\beta_1 + \beta_2 - 2| \\ &= |-1-2| \\ &> 2,\end{aligned}$$

we obtain that at least one of $\beta_1 - 1, \beta_2 - 1$ has absolute value larger than 1. Consequently, $|(\beta_i - 1)\alpha| > |\alpha|$ holds for some $1 \leq i \leq 2$. This proves the Claim. □

If $P(x)$ has a nonzero root in $\mathbb{C}$, then applying the above Claim to a root of $P(x)$ of largest absolute value, we would obtain a contradiction. Hence, $P(x)$ is equal to $cx^n$ for some $c \in \mathbb{R}$ and an integer $n \geq 0$. Moreover, any polynomial of this form satisfies the given condition. Consequently, these are all the polynomials satisfying the given condition. ■

**Remark.** Note that proving $|(\beta_1 - 1)(\beta_2 - 1)| > 1$, in order to conclude that at least one of $\beta_1 - 1, \beta_2 - 1$ has absolute value larger than 1, does not seem to work. Moreover, $|(\beta_1 - 1)(\beta_2 - 1)|$ is smaller than $\frac{1}{2}(|\beta_1 - 1| + |\beta_2 - 1|)$. **Unsurprisingly**, a lower bound for the bigger quantity can easily be obtained.

**Example 1.6** (IMOSL 1979 Bulgaria)**.** Find all polynomials $f(x)$ with real coefficients satisfying

$$f(x)f(2x^2) = f(2x^3 + x).$$

**Solution 6.** Note that $0, 1$ are the only constant polynomials satisfying the given condition.

Let $f(x)$ be a nonconstant polynomial with real coefficients satisfying the given condition.

**Claim** — The roots of $f(x)$ in $\mathbb{C}$ are of absolute value at most 1.

*Proof of the Claim.* If $\alpha$ is a root of $f(x)$ in $\mathbb{C}$, then $2\alpha^3 + \alpha$ is a root of $f(x)$. If $|\alpha| > 1$, then

$$|2\alpha^3 + \alpha| \geq 2|\alpha|^3 - |\alpha| > |\alpha|$$

holds. If some root of $f(x)$ has absolute value larger than 1, then taking $\alpha$ to be a root of $f(x)$ with largest absolute value, we would obtain a contradiction. This proves the Claim. □

**Claim** — The equality $f(0) = 1$ holds.

*Proof of the Claim.* Substituting $x = 0$, it follows that $f(0)^2 = f(0)$, and hence $f(0) = 0$ or $f(0) = 1$.

On the contrary, suppose $f(0) = 0$ holds. Write $f(x) = x^k g(x)$ where $k$ is a positive integer, and $g(x)$ is a polynomial with real coefficients with $g(0) \neq 0$. The given condition translates to

$$x^k(2x^2)^k g(x)g(2x^2) = (2x^3 + x)^k g(2x^3 + x),$$

which yields

$$(2x^2)^k g(x)g(2x^2) = (2x^2 + 1)^k g(2x^3 + x).$$

Substituting $x = 0$, we obtain $g(0) = 0$, which is impossible. This proves the Claim. □

By the above Claim, the product of the absolute values of the roots of $f(x)$ is equal to 1. By the first Claim, these absolute values are at most 1. It follows that the roots of $f(x)$ are of absolute value 1.

Let $\alpha$ be a root of $f(x)$. Note that $2\alpha^3 + \alpha$ is also a root of $f(x)$, and we have that

$$|\alpha| = |2\alpha^3 + \alpha| = 1,$$

which yields $|2\alpha^2 + 1| = |\alpha| = 1$. This gives

$$(2\alpha^2 + 1)(2\overline{\alpha}^2 + 1) = 1.$$

Combining it with $|\alpha| = 1$, we obtain

$$\alpha^2 + \overline{\alpha}^2 = -2.$$

Since $|\alpha| = 1$, it follows that $\alpha^2 = -1$, and hence, $\alpha = i$ or $\alpha = -i$. So, the polynomial $f(x)$ is equal to $c(x + i)^a(x - i)^b$ for some $c \in \mathbb{C} \setminus \{0\}$, and some nonnegative integers $a, b$. Since $f(x)$ has real coefficients, it follows that $a = b$, and $c$ lies in $\mathbb{R}$. This gives that $f(x) = c(x^2 + 1)^a$. Using $f(0) = 1$, we obtain $c = 1$, and hence $f(x) = (x^2 + 1)^a$.

Note that if $g(x)$ denotes the polynomial $(x^2 + 1)^k$, where $k$ is a positive integer, then

$$\begin{aligned} g(x)g(2x^2) &= \big((x^2 + 1)((2x^2)^2 + 1)\big)^k \\ &= \big((4x^6 + 4x^4 + x^2 + 1)\big)^k \\ &= \big(((2x^3 + x)^2 + 1)\big)^k \\ &= g(2x^3 + x). \end{aligned}$$

We conclude that the polynomials satisfying the given condition are precisely the constant polynomial $0, 1$, and the polynomials of the form $(x^2 + 1)^k$ for some positive integer $k$. ■

# References

[**Goy21**] Rohan Goyal. "Polynomials". Available at https://www.dropbox.com/s/yo31nat6z5ggaue/Polynomials.pdf?dl=0. 2021 (cited p. 2)