Rational and irrational numbers

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Rational and irrational numbers

Example 1.1 (Moscow Math Circles)**.** Does there exist irrational numbers x, y with $x > 0$ such that x^y is rational?

Summary — Consider $\sqrt{2}^{\sqrt{2}}$.

Walkthrough —

(a) Consider $\sqrt{2}^{\sqrt{2}}$.

(b) If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done by taking $x = y = \sqrt{2}$.

(c) If $\sqrt{2}^{\sqrt{2}}$ is irrational, then can you find out suitable x, y?

Solution 1. Consider $\sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we may take $x = y = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then taking $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, we find that

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2,$$

which is a rational number. ■

Example 1.2 (India RMO 2013d P4)**.** Let x be a nonzero real number such that $x^4 + \frac{1}{x^4}$ and $x^5 + \frac{1}{x^5}$ are both rational numbers. Prove that $x + \frac{1}{x}$ is a rational number.

Summary — Consider the difference

$$\left(x^5 + \frac{1}{x^5}\right) - \left(x^4 + \frac{1}{x^4}\right)\left(x + \frac{1}{x}\right).$$

Solution 2. For a positive integer n, put $y_n = x^n + \frac{1}{x^n}$. Note that

$$y_5 = y_1 y_4 - y_3 = y_1 y_4 - y_1(y_2 - 1) = y_1(y_4 - y_2 + 1),$$

and $y_4 - y_2 + 1$ is nonzero, otherwise, we would get $y_2^2 - y_2 - 1 = 0$, that is, $y_2 = \frac{1 \pm \sqrt{5}}{2}$, which shows that $y_4 = \frac{3 \pm \sqrt{5}}{2}$, which contradicts that y_4 is rational. To show that y_1 is rational, it suffices to show that y_2 is rational. Observe that

$$y_{10} = y_2 y_8 - y_6 = y_2 y_8 - y_2 y_4 + y_2 = y_2(y_8 - y_4 + 1),$$

and $y_8 - y_4 + 1 \neq 0$ (otherwise, we would obtain $y_4^2 - y_4 - 1 = 0$, which contradicts the rationality of y_4). Since y_{10}, y_8, y_4 are rational, it follows that y_2 is rational. Using

$$y_5 = y_1(y_4 - y_2 + 1),$$

and that $y_4 - y_2 + 1$ is nonzero, we conclude that y_1 is rational. ■

Example 1.3 (All-Russian MO 2001–2002 Final stage Grade 11 P1)**.** Real numbers x and y are such that $x^p + y^q$ is rational for any different odd primes p, q. Show that x and y are rational.

Solution 3. The given condition implies that for any three distinct primes p, q, r,

$$x^p - x^q, x^q - x^r$$

are rational since

$$x^p - x^q = (x^p + y^r) - (x^q + y^r), x^q - x^r = (x^q + y^p) - (x^r + y^p).$$

It follows that

$$a = x^7 - x^5, b = x^5 - x^3$$

are rational. If $b = 0$, then $x = 0$ or $x = \pm 1$, and hence x is rational. If $b \neq 0$, then note that $x^2 = \frac{a}{b}$, which shows that x^2 is rational. Observing that

$$b = x^2(x^2 - 1)x,$$

it follows that if $b \neq 0$, then x is rational. The rationality of y follows similarly. ■

Example 1.4 (British Mathematical Olympiad Round 1 2004/5 P5)**.** Let S be a set of rational numbers with the following properties:

1. $\frac{1}{2} \in S$,
2. If $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.

Prove that S contains all rational numbers in the interval $0 < x < 1$.

Walkthrough —

(a) Since $\frac{1}{2}$ lies in S, by the second condition, it follows that $\frac{2}{3}$ lies in S and so does $\frac{1}{3}$.

(b) Taking $x = \frac{1}{3}$, it follows that
$$\frac{3}{4}, \frac{1}{4}$$
lie in S. **Note that we have showed that S contains all the rationals between 0 and 1 with denominator at most 4.**

(c) Taking $x = \frac{2}{3}$, it follows that
$$\frac{2}{5}, \frac{3}{5}$$
lie in S. We are **not in a position** to conclude that S contains all the rationals between 0 and 1 with denominator at most 5.

(d) Taking $x = \frac{1}{4}$, it follows that
$$\frac{1}{5}, \frac{4}{5}$$
lie in S. It follows that S contains all the rationals between 0 and 1 with denominator at most 5.

(e) Does the above provide any insight to conclude that S contains all the rationals between 0 and 1? For instance, can one expect the following (and then prove, or realize that it is false, or argue along different lines)?

> For a rational number x lying in S, the rationals
> $$\frac{1}{x+1}, \frac{x}{x+1}$$
> have denominators larger[a] than that of x.
>
> [a]Often, while being naive, one takes the liberty to write **larger** to mean **no smaller**, that is, **greater than or equal to**. But this is **NOT allowed** while writing down a solution.

Or, stated in a different way,

> A rational number lying in $(0, 1)$ can be obtained from a rational number lying in $(0, 1)$ with smaller denominator by applying one of the maps
> $$x \mapsto \frac{1}{x+1}, x \mapsto \frac{x}{x+1}.$$

Solution 4. It suffices to establish the following.

Claim — For any integer $k \geq 2$, all the rationals lying in $(0, 1)$ with denominators not exceeding k lie in S, that is, we have

$$\left\{\frac{1}{\ell}, \frac{2}{\ell}, \dots, \frac{\ell-2}{\ell}, \frac{\ell-1}{\ell}\right\} \subseteq S \quad \text{for all } 2 \leq \ell \leq k. \tag{1}$$

Proof of the Claim. Eq. (1) holds for $k = 2$ from condition (1). Suppose Eq. (1) holds for $k = n - 1$ for some integer $n \geq 3$. Let m be an integer satisfying $1 \leq m < n$. Using the induction hypothesis, we will show that $\frac{m}{n}$ lies in S. Note that for $0 < x < 1$, the inequalities

$$0 < \frac{x}{x+1} < \frac{1}{2}, \frac{1}{2} < \frac{1}{x+1} < 1$$

hold. Using Condition (1), it follows that $\frac{m}{n}$ lies in S if $\frac{m}{n} = \frac{1}{2}$. If $0 < \frac{m}{n} < \frac{1}{2}$, then

$$\frac{x}{x+1} = \frac{m}{n}$$

holds for $x = \frac{m}{n-m}$, which is a rational number lying in $(0, 1)$ with denominator $\leq n - 1$, and by induction hypothesis, the set S contains $\frac{m}{n}$. Moreover, if $\frac{1}{2} < \frac{m}{n} < 1$, then

$$\frac{1}{x+1} = \frac{m}{n}$$

holds for $x = \frac{n-m}{m}$, which is a rational number lying in $(0, 1)$ with denominator $\leq n - 1$, and by induction hypothesis, the set S contains $\frac{m}{n}$. We conclude that for any integer $n \geq 3$, Eq. (1) holds for $k = n$ if it holds for $k = n - 1$. □

■

Example 1.5 (Junior Balkan MO TST 1999)**.** Let S be a set of rational numbers with the following properties:

1. $\frac{1}{2} \in S$,
2. If $x \in S$, then both $\frac{x}{2} \in S$ and $\frac{1}{x+1} \in S$.

Prove that S contains all the rational numbers from the interval $(0, 1)$.

Walkthrough —

(a) Taking $x = \frac{1}{2}$, it follows that S contains $\frac{2}{3}$, and hence it also contains $\frac{1}{3}$.

(b) Taking $x = \frac{1}{2}$, it follows that S contains $\frac{1}{4}$. Next, taking $x = \frac{1}{3}$, we obtain that S contains $\frac{3}{4}$.

(c) Applying the map $x \mapsto \frac{1}{x+1}$ to $x = \frac{2}{3}, \frac{1}{4}$, it follows that S contains $\frac{3}{5}, \frac{4}{5}$. Since S contains $\frac{4}{5}$, the set S also contains $\frac{2}{5}, \frac{1}{5}$.

(d) Applying $x \mapsto \frac{1}{x+1}$ to $x = \frac{1}{5}$, it follows that S contains $\frac{5}{6}$. Note that S contains $\frac{4}{6} = \frac{2}{3}, \frac{3}{6} = \frac{1}{2}, \frac{2}{6} = \frac{1}{3}$. It also follows that S contains $\frac{1}{6}$.

(e) Does the above provide any insight into the problem? Can one expect the following?

> The rationals lying in $(\frac{1}{2}, 1)$ can be obtained by applying the map $x \mapsto \frac{1}{x+1}$ to the rationals lying in $(0, 1)$ with small denominators. Moreover, a rational number r lying in $(0, \frac{1}{2})$ can be obtained by applying the map $x \mapsto \frac{x}{2}$ to the rationals lying in $(\frac{1}{2}, 1)$, more specifically, to those rationals with denominators at most the denominator of r.

Solution 5. It suffices to establish the following.

Claim — For any integer $k \geq 2$, all the rationals lying in $(0, 1)$ with denominators not exceeding k lie in S, that is, we have

$$\left\{\frac{1}{\ell}, \frac{2}{\ell}, \dots, \frac{\ell-2}{\ell}, \frac{\ell-1}{\ell}\right\} \subseteq S \quad \text{for all } 2 \leq \ell \leq k. \tag{2}$$

Proof of the Claim. Note that Eq. (2) holds for $k = 2$ by hypothesis. Suppose Eq. (2) holds for $k = n - 1$ where $n \geq 3$ is an integer. Let m be an integer satisfying $1 \leq m < n$. Using the induction hypothesis, we will show that $\frac{m}{n}$ lies in S. Note that for $0 < x < 1$, the inequalities

$$\frac{1}{2} < \frac{x}{x+1} < 1$$

hold. If $\frac{m}{n}$ is equal to $\frac{1}{2}$, then it lies in S by hypothesis. If $\frac{m}{n}$ lies in $(\frac{1}{2}, 1)$, then

$$\frac{1}{x+1} = \frac{m}{n}$$

holds for $x = \frac{n-m}{m}$, which is a rational number lying in $(0, 1)$ with denominator $\leq n - 1$, and by induction hypothesis, the set S contains $\frac{m}{n}$. If $\frac{m}{n}$ lies in $(0, \frac{1}{2})$, then the rational number $\frac{2m}{n}$ lies in $(\frac{1}{2}, 1)$, and if it has denominator $< n$ (when expressed in its least form), then it is an element of S by the induction hypothesis, and if it has denominator equal to n (when expressed in its least form), then by the above argument, it lies in S, and consequently, S contains $\frac{m}{n}$. We conclude that for any integer $n \geq 3$, Eq. (2) holds for $k = n$ if it holds for $k = n - 1$. □

■