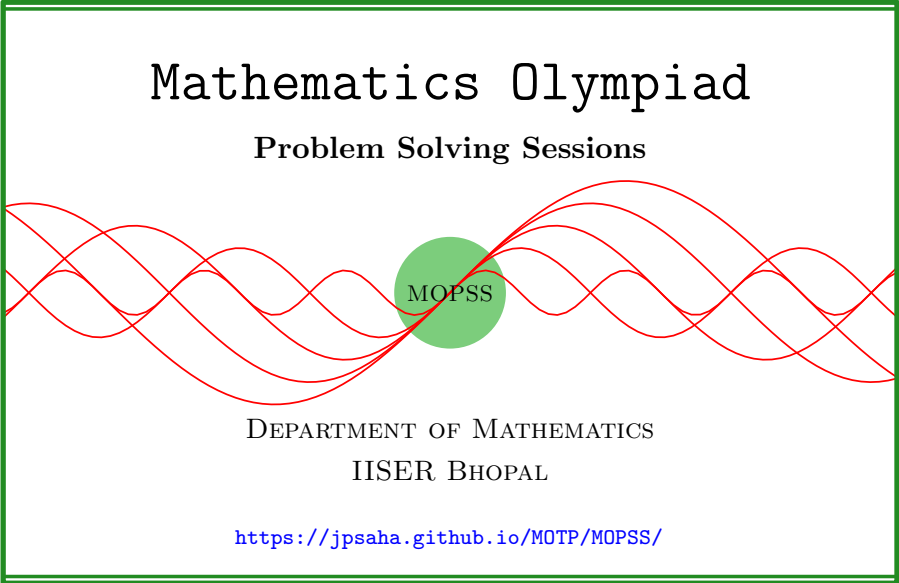


# Quartics

MOPSS

21 June 2024

The logo is enclosed in a double green border. It features the text "Mathematics Olympiad" in a large, black, serif font at the top, followed by "Problem Solving Sessions" in a smaller, black, serif font. Below this is a decorative horizontal line consisting of several overlapping red wavy lines. In the center of this line is a green circle containing the text "MOPSS" in white. At the bottom of the logo, the text "DEPARTMENT OF MATHEMATICS" and "IISER BHOPAL" is written in a black, serif font, with a URL <https://jpsaha.github.io/MOTP/MOPSS/> in blue below it.

Mathematics Olympiad  
Problem Solving Sessions

MOPSS

DEPARTMENT OF MATHEMATICS  
IISER BHOPAL

<https://jpsaha.github.io/MOTP/MOPSS/>

## Suggested readings

- **Evan Chen's**
  - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
  - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- **Evan Chen** discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

## List of problems and examples

1.1	Example (USAMO 1977 P3)	2
1.2	Example (India RMO 2012a P6)	3
1.3	Example (USAMO 2014 P1)	3

## §1 Quartics

**Example 1.1** (USAMO 1977 P3). [AE11, Problem 1.94] Let  $a$  and  $b$  be two of the roots of the polynomial  $X^4 + X^3 - 1$ . Prove that  $ab$  is a root of the polynomial  $X^6 + X^4 + X^3 - X^2 - 1$ .

**Solution 1.** Let  $c$  and  $d$  denote the remaining roots of  $X^4 + X^3 - 1$ . Using Viète's relations, we obtain

$$\begin{aligned} a + b + c + d &= -1, \\ ab + ac + ad + bc + bd + cd &= 0, \\ abc + abd + acd + bcd &= 0, \\ abcd &= -1. \end{aligned}$$

Writing the above in terms of

$$\begin{aligned} s &= a + b, \\ s' &= c + d, \\ p &= ab, \\ p' &= cd, \end{aligned}$$

we get

$$s + s' = -1, \quad p + p' + ss' = 0, \quad ps' + p's = 0, \quad pp' = -1.$$

This gives

$$p' = -\frac{1}{p}, \quad s' = -1 - s.$$

Combining them with the above yields

$$p - \frac{1}{p} - s^2 - s = 0, \quad p(-1 - s) - \frac{s}{p} = 0.$$

This shows that

$$s = -\frac{p^2}{p^2 + 1},$$

and consequently,

$$p - \frac{1}{p} - \frac{p^4}{(p^2 + 1)^2} + \frac{p^2}{p^2 + 1} = 0,$$

which is equivalent to  $p^6 + p^4 + p^3 - p^2 - 1 = 0$ . We conclude that  $p = ab$  is a root of the polynomial  $X^6 + X^4 + X^3 - X^2 - 1$ . ■

**Example 1.2 (India RMO 2012a P6).** Let  $a$  and  $b$  be real numbers such that  $a \neq 0$ . Prove that not all the roots of  $ax^4 + bx^3 + x^2 + x + 1 = 0$  can be real.

**Solution 2.** Since  $a$  is nonzero, the polynomial  $ax^4 + bx^3 + x^2 + x + 1$  has four roots in  $\mathbb{C}$ . Denote these roots by  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ . Since  $\alpha_1\alpha_2\alpha_3\alpha_4$  is equal to  $\frac{1}{a}$ , it follows that the roots  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  are nonzero. Note that the reciprocals of  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  are the roots of the polynomial  $x^4 + x^3 + x^2 + bx + a$ . This gives

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} = -1, \quad \sum_{1 \leq i < j \leq 4} \frac{1}{\alpha_i \alpha_j} = 1,$$

which yields

$$\begin{aligned} \frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{1}{\alpha_4^2} &= \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} \right)^2 - 2 \sum_{1 \leq i < j \leq 4} \frac{1}{\alpha_i \alpha_j} \\ &= -1. \end{aligned}$$

This shows that not all of  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  are real. ■

**Example 1.3 (USAMO 2014 P1).** Let  $a, b, c, d$  be real numbers such that  $b - d \geq 5$  and all zeros  $x_1, x_2, x_3, x_4$  of the polynomial  $P(x) = x^4 + ax^3 + bx^2 + cx + d$  are real. Find the smallest value the product

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$$

can take.

See also ??, [India Pre-RMO 2012 P17](#).

**Solution 3.** Note that

$$\begin{aligned} (x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) &= P(i)P(-i) \\ &= (1 - b + d)^2 + (a - c)^2 \\ &= (b - d - 1)^2 + (a - c)^2 \\ &\geq 16. \end{aligned}$$

Taking  $a = c$  and  $b = 5, d = 0$ , we obtain

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = 16.$$

Hence, the smallest value the product  $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$  can take is equal to 16. ■

## References

- [AE11] TITU ANDREESCU and BOGDAN ENESCU. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)