Quartics

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Quartics

Example 1.1 (USAMO 1977 P3)**.** [**AE11**, Problem 1.94] Let a and b be two of the roots of the polynomial $X^4 + X^3 - 1$. Prove that ab is a root of the polynomial $X^6 + X^4 + X^3 - X^2 - 1$.

Solution 1. Let c and d denote the remaining roots of $X^4 + X^3 - 1$. Using Viète's relations, we obtain

$$\begin{aligned} a + b + c + d &= -1, \\ ab + ac + ad + bc + bd + cd &= 0, \\ abc + abd + acd + bcd &= 0, \\ abcd &= -1. \end{aligned}$$

Writing the above in terms of

$$\begin{aligned} s &= a + b, \\ s' &= c + d, \\ p &= ab, \\ p' &= cd, \end{aligned}$$

we get

$$s + s' = -1, \quad p + p' + ss' = 0, \quad ps' + p's = 0, \quad pp' = -1.$$

This gives

$$p' = -\frac{1}{p}, s' = -1 - s.$$

Combining them with the above yields

$$p - \frac{1}{p} - s^2 - s = 0, \quad p(-1 - s) - \frac{s}{p} = 0.$$

This shows that

$$s = -\frac{p^2}{p^2 + 1},$$

and consequently,

$$p - \frac{1}{p} - \frac{p^4}{(p^2+1)^2} + \frac{p^2}{p^2+1} = 0,$$

which is equivalent to $p^6 + p^4 + p^3 - p^2 - 1 = 0$. We conclude that $p = ab$ is a root of the polynomial $X^6 + X^4 + X^3 - X^2 - 1$. ■

Example 1.2 (India RMO 2012a P6)**.** Let a and b be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

Solution 2. Since a is nonzero, the polynomial $ax^4 + bx^3 + x^2 + x + 1$ has four roots in $\mathbb{C}$. Denote these roots by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Since $\alpha_1\alpha_2\alpha_3\alpha_4$ is equal to $\frac{1}{a}$, it follows that the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are nonzero. Note that the reciprocals of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the polynomial $x^4 + x^3 + x^2 + bx + a$. This gives

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} = -1, \sum_{1 \leq i < j \leq 4} \frac{1}{\alpha_i \alpha_j} = 1,$$

which yields

$$\begin{aligned} \frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{1}{\alpha_4^2} &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4}\right)^2 - 2 \sum_{1 \leq i < j \leq 4} \frac{1}{\alpha_i \alpha_j} \\ &= -1. \end{aligned}$$

This shows that not all of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real. ■

Example 1.3 (USAMO 2014 P1)**.** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$$

can take.

See also **??**, India Pre-RMO 2012 P17.

Solution 3. Note that

$$\begin{aligned} (x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) &= P(i)P(-i) \\ &= (1 - b + d)^2 + (a - c)^2 \\ &= (b - d - 1)^2 + (a - c)^2 \\ &\geq 16. \end{aligned}$$

Taking $a = c$ and $b = 5, d = 0$, we obtain

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = 16.$$

Hence, the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take is equal to 16. ■

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. isbn: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)