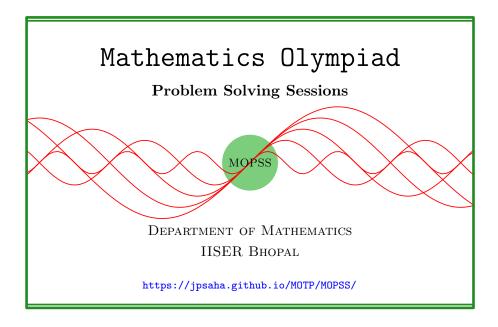
Quartics

MOPSS

 $20 \ \mathrm{March} \ 2025$



Suggested readings

- Evan Chen's
 - advice On reading solutions, available at https://blog.evanchen. cc/2017/03/06/on-reading-solutions/.
 - Advice for writing proofs/Remarks on English, available at https: //web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Quartics

Example 1.1 (USAMO 1977 P3). [AE11, Problem 1.94] Let *a* and *b* be two of the roots of the polynomial $X^4 + X^3 - 1$. Prove that *ab* is a root of the polynomial $X^6 + X^4 + X^3 - X^2 - 1$.

Solution 1. Let *c* and *d* denote the remaining roots of $X^4 + X^3 - 1$. Using Viète's relations, we obtain

$$a + b + c + d = -1,$$

$$ab + ac + ad + bc + bd + cd = 0,$$

$$abc + abd + acd + bcd = 0,$$

$$abcd = -1.$$

Writing the above in terms of

$$s = a + b,$$

$$s' = c + d,$$

$$p = ab,$$

$$p' = cd,$$

we get

$$s + s' = -1$$
, $p + p' + ss' = 0$, $ps' + p's = 0$, $pp' = -1$.

This gives

$$p' = -\frac{1}{p}, s' = -1 - s.$$

Combining them with the above yields

$$p - \frac{1}{p} - s^2 - s = 0, \quad p(-1-s) - \frac{s}{p} = 0.$$

This shows that

$$s = -\frac{p^2}{p^2 + 1},$$

and consequently,

$$p - \frac{1}{p} - \frac{p^4}{(p^2 + 1)^2} + \frac{p^2}{p^2 + 1} = 0,$$

which is equivalent to $p^6 + p^4 + p^3 - p^2 - 1 = 0$. We conclude that p = ab is a root of the polynomial $X^6 + X^4 + X^3 - X^2 - 1$.

Example 1.2 (India RMO 2012a P6). Let *a* and *b* be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

Solution 2. Since *a* is nonzero, the polynomial $ax^4 + bx^3 + x^2 + x + 1$ has four roots in \mathbb{C} . Denote these roots by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Since $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ is equal to $\frac{1}{a}$, it follows that the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are nonzero. Note that the reciprocals of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the polynomial $x^4 + x^3 + x^2 + bx + a$. This gives

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} = -1, \sum_{1 \le i < j \le 4} \frac{1}{\alpha_i \alpha_j} = 1,$$

which yields

$$\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{1}{\alpha_4^2} = \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4}\right)^2 - 2\sum_{1 \le i < j \le 4} \frac{1}{\alpha_i \alpha_j}$$
$$= -1.$$

This shows that not all of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real.

Example 1.3 (USAMO 2014 P1). Let a, b, c, d be real numbers such that $b-d \ge 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product

$$(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$$

can take.

See also ??, India Pre-RMO 2012 P17.

Solution 3. Note that

$$\begin{aligned} (x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1) &= P(i)P(-i) \\ &= (1-b+d)^2 + (a-c)^2 \\ &= (b-d-1)^2 + (a-c)^2 \\ &> 16. \end{aligned}$$

Taking a = c and b = 5, d = 0, we obtain

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = 16.$$

Hence, the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take is equal to 16.

References

[AE11] TITU ANDREESCU and BOGDAN ENESCU. Mathematical Olympiad treasures. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)