Quadratic polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Quadratic polynomials

Example 1.1 (Hungary MO 2001/02, Grades 11 and 12 — technical schools, P2)**.** Consider the following 2000 equations:

$$\begin{aligned} 1x^2 + 2 \cdot 2x + 3 &= 0 \\ 2x^2 + 2 \cdot 3x + 4 &= 0 \\ 3x^2 + 2 \cdot 4x + 5 &= 0 \\ &\vdots \\ 2000x^2 + 2 \cdot 2001x + 2002 &= 0. \end{aligned}$$

For each equation, consider the product of the sum of the real roots and the sum of their reciprocals (if it exists). What is the product of these products?

Solution 1. Note that a quadratic polynomial with nonzero constant term has nonzero roots, and hence the reciprocals of its roots exist. Moreover, if α, β denote the roots of a quadratic polynomial $ax^2 + bx + c$ with $ac \neq 0$, then $1/\alpha, 1/\beta$ are the roots of quadratic polynomial $cx^2 + bx + a$. Note that

$$\alpha + \beta = -\frac{b}{a}, \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}$$

Since

$$(2(n+1))^2 - 4n(n+2) = 4n^2 + 8n + 4 - 4n^2 - 8n = 4 \geq 0,$$

it follows that the given 2000 equations have real roots. The product of the products considered is equal to

$$\frac{2^2 \cdot 2^2}{1 \cdot 3} \times \frac{2^2 \cdot 3^2}{2 \cdot 4} \times \frac{2^2 \cdot 4^2}{3 \cdot 5} \times \frac{2^2 \cdot 5^2}{4 \cdot 6} \times \cdots \times \frac{2^2 \cdot 2000^2}{1999 \cdot 2001} \times \frac{2^2 \cdot 2001^2}{2000 \cdot 2002}$$

$$
\begin{aligned}
&= \frac{4^{2000}}{1\cdot 2}\frac{2^2\cdot 3^2\cdot\dots\cdot 2001^2}{3^2\cdot 4^2\cdot\dots\cdot 2000^2}\frac{1}{2001\cdot 2002}\\
&= \frac{4^{2000}\cdot 2^2\cdot 2001^2}{1\cdot 2\cdot 2001\cdot 2002}\\
&= \frac{2001}{1001}\times 2^{4000}.
\end{aligned}
$$

■

Example 1.2 (Canada CMO 1971 P4)**.** Determine all real numbers a such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common.

Solution 2. Let a be a real nuber such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common. Let α denote a common root of these polynomials. The equations

$$\alpha^2+a\alpha+1=0, \alpha^2+\alpha+a=0$$

yield

$$(a-1)\alpha = a-1.$$

If $a\neq 1$, then $\alpha=1$ and hence $a=-2$. This proves that $a=1,-2$.

If $a=1$, then the given polynomials have at least one root in common. If $a=-2$, then the given polynomials vanish at 1.

We conclude that $a=1,-2$ are precisely all the real numbers such that the given polynomials have at least one common root. ■

Example 1.3 (India RMO 2003 P6)**.** Find all real numbers a for which the equation

$$x^2+(a-2)x+1=3|x|$$

has exactly three distinct real solutions for x.

Solution 3. Let a be a real number such that $x^2+(a-2)x+1=3|x|$ has exactly three distinct real solutions. Note that the equation

$$(x^2+(a-2)x+1-3x)(x^2+(a-2)x+1+3x)=0$$

also has exactly three distinct real solutions. It follows that the discriminant of one of the polynomials $x^2+(a-2)x+1-3x$, $x^2+(a-2)x+1+3x$ vanishes, and the discriminant of the other is positive. The discriminants of these polynomials are

$$
\begin{aligned}
(a-5)^2-4 &= a^2-10a+21\\
&= (a-3)(a-7),
\end{aligned}
$$

$$\begin{aligned}(a+1)^2-4 &= a^2+2a-3\\ &= (a+3)(a-1)\end{aligned}$$

respectively. It follows that a does not belong to $(-\infty,-3)\cup(-3,1)\cup(1,3)\cup(3,7)\cup(7,\infty)$, or equivalently, a belongs to $\{-3,1,3,7\}$.

Let us determine whether the given equation has three distinct real roots if a lies in $\{-3,1,3,7\}$. Let us assume that a lies in $\{-3,1,3,7\}$. Note that then one of $(a-3)(a-7),(a+3)(a-1)$ vanishes and another is positive, and consequently, one of the polynomials

$$x^2+(a-2)x+1-3x, x^2+(a-2)x+1+3x$$

have distinct real roots, and the other has a double root, which is a real number. Observe that if $x^2+(a-2)x+1-3x$ has a double root, then that root is equal to $-\frac{a-5}{2}$. Note that

$$\begin{aligned}\left(\frac{a-5}{2}\right)^2-(a+1)\left(\frac{a-5}{2}\right)+1 &= \frac{1}{4}\left(a^2-10a+25-2a^2+8a+10+4\right)\\ &= \frac{1}{4}\left(-a^2-2a+39\right),\end{aligned}$$

which has a negative discriminant. This shows that if $x^2+(a-2)x+1-3x$ has a real double root, then that cannot be a zero of $x^2+(a-2)x+1+3x$. Also note that

$$\begin{aligned}\left(\frac{a+1}{2}\right)^2-(a-5)\frac{a+1}{2}+1 &= \frac{1}{4}\left(a^2+2a+1-2a^2+8a+10+4\right)\\ &= \frac{1}{4}\left(-a^2+10a+15\right),\end{aligned}$$

whose roots are not integers. Using that a is an integer, it follows that if $x^2+(a-2)x+1+3x$ has a real double root, then that cannot be a root of $x^2+(a-2)x+1-3x$. We conclude that if a lies in $\{-3,1,3,7\}$, then

$$(x^2+(a-2)x+1-3x)(x^2+(a-2)x+1+3x)=0$$

has exactly three distinct real solutions, or equivalently, the equation

$$x^2+(a-2)x+1=3|x|$$

has exactly three distinct real solutions for x.

So the required real numbers are $a=-3,1,3,7$. ∎

Example 1.4 (All-Russian MO 2007 Grade 8 P1)**.** If a,b,c are real numbers, show that at least one of the equations

$$x^2+(a-b)x+(b-c)=0,$$

$$x^2 + (b-c)x + (c-a) = 0,$$
$$x^2 + (c-a)x + (a-b) = 0$$

has a real solution.

Solution 4. The sum of the discriminants of the above quadratic polynomials is

$$(a-b)^2 - 4(b-c) + (b-c)^2 - 4(c-a) + (c-a)^2 - 4(a-b)$$
$$= (a-b)^2 + (b-c)^2 + (c-a)^2,$$

which is positive if not all of a, b, c are equal. Consequently, if not all of the three real numbers a, b, c are equal, then at least one of the quadratic polynomials

$$x^2 + (a-b)x + (b-c), x^2 + (b-c)x + (c-a), x^2 + (c-a)x + (a-b)$$

has positive discriminant, and hence admits real solutions. Moreover, if all of a, b, c are equal, at least one (in fact, all) of the above polynomials admits a real root. ■

Example 1.5 (India RMO 2007 P3)**.** Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

Solution 5. Let a, b be real numbers such that for any root α of $x^2+ax+b = 0$, $\alpha^2 - 2$ is also a root. Denote the roots of $x^2 + ax + b$ by α, β. There are the following possibilities.

(1) $\alpha^2 - 2 = \alpha$, $\beta^2 - 2 = \beta$,

(2) $\alpha^2 - 2 = \beta$, $\beta^2 - 2 = \alpha$,

(3) $\alpha^2 - 2 = \beta^2 - 2 = \alpha$,

(4) $\alpha^2 - 2 = \beta^2 - 2 = \beta$.

If $\alpha = \beta$, then these four cases are equivalent to

$$\alpha^2 - 2 = \beta^2 - 2 = \alpha = \beta,$$

which shows that α is equal to 2 or -1, and hence (a, b) is equal to $(-4, 4)$ or $(2, 1)$.

It remains to consider the case that $\alpha \neq \beta$, which we assume from now on.

In Case (1), α, β satisfy the equation $X^2 - X - 2 = 0$. So (α, β) is equal to $(2, -1)$ or $(-1, 2)$, and hence (a, b) is equal to $(-1, -2)$.

In Case (2), we have $\alpha^2 - \beta^2 = \beta - \alpha$, which gives $\alpha + \beta = -1$ (since $\alpha \neq \beta$). So

$$\alpha\beta = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(\alpha^2+\beta^2) = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(\alpha+\beta+4) = \frac{1}{2} - \frac{3}{2} = -1.$$

This shows that α, β are roots of the quadratic polynomial $x^2 + x - 1$, and hence, (a, b) is equal to $(1, -1)$.

In Case (3), note that α is equal to 2 or -1. Using $\beta^2 = 2 + \alpha$ and $\alpha \neq \beta$, it follows that (α, β) is equal to $(2, -2)$ or $(-1, 1)$, and hence (a, b) is equal to $(0, -4)$ or $(0, -1)$.

Similarly, in Case (4), (α, β) is equal to $(-2, 2)$ or $(-1, 1)$, which shows (a, b) is equal to $(0, -4)$ or $(0, -1)$.

So (a, b) is equal to one of $(-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1)$.

Moreover, if (a, b) is equal to any of these six pairs, then it can be checked that for any root α of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root.

We conclude that all the required pairs are $(-4, 4)$, $(2, 1)$, $(-1, -2)$, $(1, -1)$, $(0, -4)$, $(0, -1)$. ■

Example 1.6 (India RMO 2010 P2)**.** Let

$$P_1(x) = ax^2 - bx - c, P_2(x) = bx^2 - cx - a, P_3(x) = cx^2 - ax - b$$

be three quadratic polynomials where a, b, c are nonzero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

Solution 6. Since $P_1(\alpha), P_2(\alpha)$ are equal, we get

$$(a-b)\alpha^2 = (b-c)\alpha + (c-a),$$

which gives

$$(a-b)(\alpha^2+1) = (b-c)(\alpha-1).$$

Similarly, using $P_2(\alpha) = P_3(\alpha)$, we obtain

$$(b-c)(\alpha^2+1) = (c-a)(\alpha-1).$$

If $\alpha = 1$, then it follows that $a = b = c$. Henceforth, let us assume that $\alpha \neq 1$. Then the above yields

$$(a-b)(c-a) = (b-c)^2.$$

Using a similar argument as above, it follows that

$$(b-c)(a-b) = (c-a)^2, (c-a)(b-c) = (a-b)^2.$$

Adding these equations, we obtain

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Since a, b, c are real, it follows that $a = b = c$. This completes the proof. ■

Example 1.7 (India RMO 2012f P1)**.** Find nonzero real numbers a, b such that $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are three distinct polynomials with a common root.

Solution 7. Let a, b be real numbers such that $x^2+ax+b, x^2+x+ab, ax^2+x+b$ are three distinct polynomials with a common root $\alpha \in \mathbb{C}$. We obtain

$$\alpha^2 + a\alpha + b = \alpha^2 + \alpha + ab = a\alpha^2 + \alpha + b = 0,$$

which gives $a\alpha + b = \alpha + ab$, that is, $(a-1)(\alpha - b) = 0$. Since $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are distinct, it follows that $a \neq 1$. This shows that $\alpha = b$. Since the polynomials $x^2 + x + ab, ax^2 + x + b$ vanish at $x = \alpha = b$, we obtain

$$b(a + b + 1) = b(ab + 2) = 0.$$

Using b is nonzero, we get $a + b + 1 = ab + 2 = 0$. Note that $1 + ab - a - b = 0$. Since $a \neq 1$, we obtain $b = 1$, which combined with $ab + 2 = 0$ implies that $a = -2$.

Also note that for $a = -2, b = 1$, the given polynomials are equal to

$$x^2 - 2x + 1, x^2 + x - 2, -2x^2 + x + 1,$$

which are all distinct and they vanish at $x = 1$.

We conclude that precisely for $(a, b) = (-2, 1)$, the given polynomials are all distinct and have a common root. ■

Example 1.8 (India RMO 2015d P2)**.** Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numbers s is not equal to t such that $P(s) = t$ and $P(t) = s$. Prove that $b - st$ is a root of the equation $x^2 + ax + b - st = 0$.

Solution 8. We have

$$s^2 + as + b = t, t^2 + at + b = s.$$

Taking their difference, we obtain $(s - t)(s + t + a + 1) = 0$, which gives $s + t + a + 1 = 0$ since $s \neq t$. Using the above, we obtain

$$s(s^2 + as + b) - t(t^2 + at + b) = 0,$$

or equivalently,

$$(s - t)(b + a(s + t) + s^2 + st + t^2) = 0.$$

Combining the above with $s + t + a + 1 = 0$ and $s \neq t$, we obtain

$$b - (s + t) - st = 0.$$

Note that

$$\begin{aligned}(b-st)^2+a(b-st)+b-st &= (b-st)(b-st+a+1)\\ &= (b-st)(s+t+a+1)\\ &= 0.\end{aligned}$$

This completes the proof. ■

Example 1.9 (India RMO 2015a P2)**.** Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a, b are real numbers. Suppose $\langle P(-1)^2, P(0)^2, P(1)^2\rangle$ is an AP of positive integers. Prove that a, b are integers.

Solution 9. Note that $P(-1)^2, P(0)^2, P(1)^2$ are equal to

$$1+a^2+b^2-2a+2b-2ab, b^2, 1+a^2+b^2+2a+2b+2ab$$

respectively. Since they form an arithmetic progression, we obtain

$$1+a^2+b^2-2a+2b-2ab+1+a^2+b^2+2a+2b+2ab = 2b^2,$$

or equivalently, $a^2+2b+1=0$. It follows that

$$b^2-2a-2ab, b^2, b^2+2a+2ab$$

form an arithmetic progression of positive integers. Note that

$$\begin{aligned}(2a+2ab)^2 &= 4a^2(b+1)^2\\ &= -4(2b+1)(b+1)^2\\ &= -4(2b^3+5b^2+4b+1)\\ &= -4((2b^2+4)b+5b^2+1).\end{aligned}$$

Since b^2 is an integer, it follows that b is rational number. Since b is rational and b^2 is an integer, it follows that b is an integer. Using $a^2+2b+1=0$, it follows that a^2 is an integer. Moreover, if $b=-1$, then a is an integer. If $b \neq -1$, then using that $2a+2ab$ is an integer, we obtain a is rational. Since a^2 is an integer and a is rational, it follows that a is an integer. This completes the proof. ■

Example 1.10 (India RMO 2015b P2)**.** Let $P(x) = x^2+ax+b$ be a quadratic polynomial where a is real and $b \neq 2$, is rational. Suppose $P(0)^2, P(1)^2, P(2)^2$ are integers, prove that a and b are integers.

Solution 10. Since b is rational and $P(0)^2 = b^2$ is an integer, it follows that b is an integer. Note that

$$P(1)^2 = (1+a+b)^2$$

$$
\begin{aligned}
&= 1 + a^2 + b^2 + 2a + 2b + 2ab, \\
P(2)^2 &= (4 + 2a + b)^2 \\
&= 16 + 4a^2 + b^2 + 16a + 8b + 4ab.
\end{aligned}
$$

Since b is an integer, the given conditions imply that $a^2+2a+2ab, 4a^2+16a+4ab$ are integers. This shows that

$$
\begin{aligned}
4a^2 + 16a + 4ab - 2(a^2 + 2a + 2ab) &= 2a^2 + 12a, \\
4a^2 + 16a + 4ab - 4(a^2 + 2a + 2ab) &= 8a - 4ab
\end{aligned}
$$

are integers. Since $b \neq 2$ and b is an integer, it follows that a is a rational number. Combining this with the fact that $2a^2 + 12a$ is rational, it follows that a is equal to $\frac{n}{2}$ for some integer n. Indeed, write $a = \frac{x}{y}$ where x, y are integers with $y \geq 1$ and $\gcd(x, y) = 1$. Note that $2\frac{x^2}{y} + 12x$ is an integer. Since x and y are relatively prime, this implies that y divides 2. Consequently, a is equal to $\frac{n}{2}$ for some integer n. Using that $2a^2 + 12a$ is an integer, we get that $\frac{n^2}{2}$ is also an integer. This shows that n is even, and hence a is an integer. This completes the proof. ■

Example 1.11 (India RMO 2015e P2). Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that $P_1(m) = P_2(n), P_2(m) = P_1(n)$. Prove that $a_1 - a_2$ is even.

Solution 11. Using $P_1(m) = P_2(n)$, we get

$$m^2 + a_1m + b_1 = n^2 + a_2n + b_2,$$

that is,

$$(m^2 - n^2) + (a_1m - a_2n) + b_1 - b_2 = 0.$$

Similarly, using $P_1(n) = P_2(m)$, we get

$$(n^2 - m^2) + (a_1n - a_2m) + b_1 - b_2 = 0.$$

This yields

$$2(m^2 - n^2) + (a_1 + a_2)(m - n) = 0.$$

Since $m \neq n$, we get $2(m + n) + a_1 + a_2 = 0$. It follows that $a_1 + a_2$ is even, and hence, so is $a_1 + a_2 - 2a_2 = a_1 - a_2$. ■