# More on polynomials

MOPSS

23 June 2024

Mathematics Olympiad

**Problem Solving Sessions**

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

## Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

# List of problems and examples



## §1 More on polynomials

**Example 1.1** (India RMO 2012e P2)**.** cf. [**GA17**, Problem 141] Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of $P$ are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

**Solution 1.** Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of $P$. Note that

$$\begin{aligned}
&\sum_{1\leq i,j\leq n, i\neq j} (\alpha_i - \alpha_j)^2 \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 2\sum_{1\leq i,j\leq n, i\neq j} \alpha_i\alpha_j \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2(n-1)\left(\sum_{1\leq i\leq n} \alpha_i\right)^2 - 4(n-1)\left(\sum_{1\leq i<j\leq n} \alpha_i\alpha_j\right) - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\binom{n}{2} \\
&= 0.
\end{aligned}$$

Since $\alpha_1, \ldots, \alpha_n$ are real, it follows that they are all equal. Using $\alpha_1 + \cdots + \alpha_n = n$, we get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1.$$

This implies that $a_i = -\binom{n}{i}$ for any $0 \leq i \leq n-1$. ■

**Solution 2.** Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of $P(x)$. Note that

$$\begin{aligned}
&(\alpha_1 - 1)^2 + (\alpha_2 - 1)^2 + \cdots + (\alpha_n - 1)^2 \\
&= \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \\
&= (\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2 - 2\sum_{1\leq i<j\leq n} \alpha_i\alpha_j + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n)
\end{aligned}$$

$$= n^2 - n(n-1) + n - 2n$$
$$= 0.$$

So the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all equal to 1. This implies that $P(x) = (x-1)^n$, and hence $a_i$ is equal to $(-1)^i \binom{n}{i}$ for any $0 \leq i \leq n$. ■

**Example 1.2** (D. A. Zvonkin, Tournament of Towns, Spring 2014, Senior, A Level, P7)**.** Consider a polynomial $P(x)$ such that

$$P(0) = 1, \quad (P(x))^2 = 1 + x + x^{100}Q(x),$$

where $Q(x)$ is also a polynomial. Prove that in the polynomial $(P(x)+1)^{100}$, the coefficient of $x^{99}$ is zero.

**Example 1.3.** [**WH96**, Problem 27] Let $p_1, \ldots, p_n$ denote $n \geq 1$ distinct integers. Show that the polynomial

$$(x-p_1)^2(x-p_2)^2 \cdots (x-p_n)^2 + 1$$

cannot be expressed as the product of two non-constant polynomials with integral coefficients.

**Solution 3.** On the contrary, let us assume that the polynomial

$$P(x) := (x-p_1)^2(x-p_2)^2 \cdots (x-p_n)^2 + 1$$

can be expressed as the product of two non-constant polynomials $f(x), g(x)$ with integral coefficients.

Let us first establish the following Claims.

> **Claim** — Replacing $f, g$ by $-f, -g$ respectively (if necessary), we may assume that $f, g$ take positive values at all real arguments.

*Proof of the Claim.* Note that the polynomial $P(x)-1$ vanishes at $x = p_1, \ldots, p_n$. Since the product of the leading coefficients of $f(x)$ and $g(x)$ is equal to the leading coefficient of $P(x)$, we may replace $f(x), g(x)$ by $-f(x), -g(x)$ respectively (if necessary) to assume that the leading coefficients of $f(x), g(x)$ are positive. Since $P = fg$ and $P$ does not have a real root, it follows that the polynomials $f, g$ do not have any real roots. At large enough real arguments, the polynomials $f, g$ take positive values. Since $f, g$ have no real roots, we conclude that they take positive values at all real arguments. □

> **Claim** — The polynomials $f, g$ are of degree $n$. Moreover, these polynomials are equal.

*Proof of the Claim.* On the contrary, let us asssume that the degrees of $f, g$ are not equal. Interchanging $f, g$ if necessary, we assume that $\deg(f) < \deg(g)$. Since the sum of the degrees of $f, g$ is equal to $2n$, it follows that $\deg(f) < n$.

For any $1 \leq i \leq n$, the integers $f(p_i), g(p_i)$ are equal to 1 or $-1$. Since $f, g$ take positive values at all real arguments, we obtain $f(p_i) = 1$ for any $1 \leq i \leq n$. This shows that the polynomial $f - 1$ has at least $n$ distinct roots. Using $\deg(f) < n$, we conclude that $f - 1$ is the zero polynomial, which is impossible since $f$ is a non-constant polynomial. Therefore, the hyothesis that the degrees of $f, g$ are not equal is not tenable. This completes the proof of the first part of the Claim.

Note that $f, g$ are polynomials of degree $n$ with equal leading coefficients. This shows that the polynomial $f(x) - g(x)$ has degree less than $n$ and it vanishes at the $n$ distinct points $p_1, \dots, p_n$. It follows that $f = g$.

□

Using the above Claim, note that

$$f(x)^2 - ((x - p_1)(x - p_2) \cdots (x - p_n))^2 = 1,$$

or equivalently,

$$\big(f(x) + (x - p_1)(x - p_2) \cdots (x - p_n)\big)\big(f(x) - (x - p_1)(x - p_2) \cdots (x - p_n)\big) = 1,$$

which implies that the polynomials

$$f(x) + (x - p_1)(x - p_2) \cdots (x - p_n), f(x) - (x - p_1)(x - p_2) \cdots (x - p_n)$$

are constant polynomials, and both of them are equal. Consequently, the polynomial $(x-p_1)(x-p_2)\cdots(x-p_n)$ is the zero polynomial, which is impossible. This shows that the hypothesis that the given polynomial can be expressed as the product of two non-constant polynomials with integral coefficients is not tenable. This completes the proof. ■

# References

[**GA17**] Răzvan Gelca and Titu Andreescu. *Putnam and beyond*. Second. Springer, Cham, 2017, pp. xviii+850. isbn: 978-3-319-58986-2; 978-3-319-58988-6. doi: 10.1007/978-3-319-58988-6. url: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 2)

[**WH96**] Kenneth S. Williams and Kenneth Hardy. *The Red Book of mathematical problems*. Corrected reprint of the ıt The Red Book: 100 practice problems for undergraduate mathematics competitions [Integer Press, Ottawa, ON, 1988]. Dover Publications, Inc., Mineola, NY, 1996, pp. x+174. isbn: 0-486-69415-1 (cited p. 3)