Lagrange interpolation

MOPSS

20 March 2025

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Lagrange interpolation

Lemma 1

Let $x_1, \ldots, x_n$ be pairwise distinct real numbers, and $y_1, \ldots, y_n$ be real numbers. Then there exists a unique polynomial $P(x)$ of **degree at most** $n-1$ having real coefficients such that $P(x_i) = y_i$ for all $1 \leq i \leq n$. Moreover, this statement also holds if the reals are replaced by rationals or complex numbers all throughout.

Proof. Note that there is at most one polynomial satisfying the required condition. Observe that the polynomial $P(x)$, defined by

$$P(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j},$$

satisfies the required condition. □

Exercise 1.1. If a polynomial of degree n takes rationals to rationals on $n+1$ points, then show that it is a rational polynomial.

Lemma 2

Let $x_1, \ldots, x_n$ be pairwise distinct real numbers, and $y_1, \ldots, y_n$ be real numbers. Then there exists a unique **monic** polynomial $P(x)$ of **degree** n having real coefficients such that $P(x_i) = y_i$ for all $1 \leq i \leq n$. Moreover, this statement also holds if the reals are replaced by rationals or complex numbers all throughout.

Proof. By the above lemma, there exists a polynomial $Q(x)$ of degree at most $n-1$ with real coefficients such that $Q(x_i) = y_i - x_i^n$ for all $1 \leq i \leq n$. Write $P(x) = x^n + Q(x)$. Note that $P(x)$ is a monic polynomial of degree n with real coefficients and $P(x_i) = y_i$ for all $1 \leq i \leq n$. □

Example 1.2. Suppose $P(x)$ is a monic polynomial of degree $n-1$ with real coefficients. Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. Show that

$$\sum_{i=1}^{n} \frac{P(a_i)}{\prod_{j\neq i}(a_j - a_i)} = 1.$$

Solution 1. For $1 \leq i \leq n$, write $y_i = P(a_i)$. Note that

$$P(x) = \sum_{i=1}^{n} y_i \prod_{j\neq i} \frac{x - a_i}{a_i - a_j}.$$

Comparing the leading coefficients, the result follows. ■

Example 1.3. Let $P(x)$ be a monic polynomial of degree n. Show that

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} P(i) = n!.$$

> **Walkthrough** — Is the above of some use?

Example 1.4 (USAMO 2002 P3). Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

> **Walkthrough** —
>
> **(a)** Let $F(x)$ be a monic polynomial of degree n with real coefficients. We would like to write
> $$2F(x) = P(x) + Q(x),$$
> where $P(x), Q(x)$ are polynomials with certain properties.
>
> **(b)** Let us take $P(x)$ to be a polynomial which changes sign very often, so that it is likely to have n real roots. To do so, choose n real numbers satisfying
> $$x_1 < x_2 < \cdots < x_n,$$
> and let $y_1, \ldots, y_n$ be real numbers (to be specified later). Apply the Lagrange interpolation formula to obtain a monic polynomial $P(x)$ satisfying $P(x_i) = y_i$ for all i.
>
> **(c)** Define the polynomial $Q(x)$ using
> $$2F(x) = P(x) + Q(x).$$
> Note that $Q(x)$ is a monic polynomial with real coefficients.

(d) Can one impose suitable conditions on $y_1, \dots, y_n$ such that $Q(x)$ changes sign often?

Solution 2. Let $F(x)$ be a monic polynomial od degree n with real coefficients. Let $x_1 < \cdots < x_n$ be real numbers. Note that there exist real numbers $y_1, \dots, y_n$ satisfying

$$(-1)^i y_i > 0, \quad (-1)^{i-1}(2F(x_i) - y_i) > 0$$

for any $1 \leq i \leq n$. Indeed, y_i's can be taken satisfying $(-1)^i y_i > |F(x_i)|$. Let $P(x)$ be a monic polynomial of degree n with real coefficients such that $P(x_i) = y_i$ for all $1 \leq i \leq n$. Let $Q(x)$ denote the polynomial such that $F(x)$ is the average of $P(x)$ and $Q(x)$. Since $F(x), P(x)$ are monic, it follows that so is $Q(x)$. For any $1 \leq i \leq n$, note that $Q(x_i) = 2F(x_i) - P(x_i) = 2F(x_i) - y_i$ holds, and hence $Q(x_i)$ has sign as that of $(-1)^i$. It follows that each of the polynomials $P(x), Q(x)$ has at least one root in each of the intervals

$$(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n).$$

Since these polynomials are of degree n with real coefficients and each of them has at least $n-1$ real roots, all of their roots are real. ■

Example 1.5 (Putnam 1968 A6)**.** Find all polynomials whose coefficients are all ± 1 and whose roots are all real.

Walkthrough —

(a) Consider the average of the squares of the roots, and show that it is small (and consequently, smaller than their geometric mean) if the polynomial has degree ≥ 4.

(b) Repeat the argument for degree three polynomials.

(c) Finding the degree one and degree two Polynomials is easy.

Solution 3. Let $P(x)$ be a polynomial of degree n having real roots. Assume that its coefficients are equal to ± 1. Denote its roots by $\alpha_1, \dots, \alpha_n$, counting multiplicities. Noting

$$\alpha_1^2 + \cdots + \alpha_n^2 = (\alpha_1 + \cdots + \alpha_n)^2 - 2 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j,$$

it follows that

$$\alpha_1^2 + \cdots + \alpha_n^2 = 3.$$

Applying the AM-GM inequality, we obtain

$$\alpha_1^2 + \cdots + \alpha_n^2 \geq n,$$

which implies that $n \leq 3$.

If $P(x)$ is a monic linear polynomial, then it is equal to one of $x-1, x+1$.

If $P(x)$ is a monic quadratic polynomial, then considering discriminants, it follows that its constant term is equal to 1, and hence $P(x)$ is equal to one of x^2+x-1, x^2-x-1.

Let us consider the case that $P(x)$ is a monic cubic polynomial, and the coefficient of x^2 in $P(x)$ is equal to 1. Note that x^3+x^2+x-1 vanishes at $x=0$. For any real number $\alpha \leq 0$,

$$\alpha^3+\alpha^2+\alpha-1 = \alpha(\alpha+1)^2 - 1 - \alpha^2 \leq -1$$

holds, which implies that any real root of x^3+x^2+x-1 is positive. Since x^3+x^2+x-1 is of odd degree, it has a real root, and since $x \mapsto x^3+x^2+x-1$ is an increasing function on the set of positive reals, it follows that x^3+x^2+x-1 has only one real root. This gives $P(x) \neq x^3+x^2+x-1$.

Let us consider the case that $P(x) = x^3+x^2-x+1$ Using

$$\begin{aligned}
P(x) &= (x^2-x)+1+x^3, \\
P(x) &= x^3+x^2+(1-x), \\
P(x) &= x^3+(x^2-x)+1,
\end{aligned}$$

it follows that $P(x)$ has no root in $[-1,\infty)$. Note that for $-1 \geq a \geq b$, we have

$$\begin{aligned}
P(a)-P(b) &= (a-b)(a^2+ab+b^2+a+b-1) \\
&= (a-b)((a^2+a)+(b^2+b)+(ab-1)) \\
&\geq 0.
\end{aligned}$$

This shows that x^3+x^2-x+1 has no real root, and hence $P(x) \neq x^3+x^2-x+1$.

Note that not all the roots of

$$x^3+x^2+x+1 = (x+1)(x^2+1)$$

are real. Consequently, if $P(x)$ is a monic cubic polynomial and the coefficient of x^2 in $P(x)$ is equal to 1, then $P(x)$ is equal to x^3+x^2-x-1.

If $P(x)$ is a monic cubic polynomial and the coefficient of x^2 in $P(x)$ is -1, then $-P(-x)$ is equal to x^3+x^2-x-1, and hence $P(x)$ is equal to x^3-x^2-x+1.

This shows that $P(x)$ is equal to one of

$$x-1, x+1, x^2+x-1, x^2-x-1, x^3+x^2-x-1, x^3-x^2-x+1.$$

Note that the discriminants of the quadratic polynomial x^2+x-1, x^2-x-1 are nonnegative, and also note that

$$\begin{aligned}
x^3+x^2-x-1 &= (x-1)(x+1)^2, \\
x^3-x^2-x+1 &= (x-1)^2(x+1).
\end{aligned}$$

Consequently, the roots of the polynomials

$$x-1, x+1, x^2+x-1, x^2-x-1, x^3+x^2-x-1, x^3-x^2-x+1$$

are all real. Hence, the required polynomials are

$$x-1, x+1, x^2+x-1, x^2-x-1, x^3+x^2-x-1, x^3-x^2-x+1,$$

$$-(x-1), -(x+1), -(x^2+x-1), -(x^2-x-1), -(x^3+x^2-x-1), -(x^3-x^2-x+1).$$

■