Inequalities

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

Contents

§1 Warm up

Example 1.1 (Nesbitt's inequality 1903, Moscow MO 1963 Grade 9, UK BMO 1976 P2, India RMO 1990 P2)**.** Let $a, b, c > 0$. Then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

This has many proofs, for instance using AM-GM, AM-HM, Cauchy-Schwarz inequality, rearrangement inequality. We present a quick proof from [**Hun08**].

Solution 1. Put

$$\alpha = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$
$$\beta = \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b},$$
$$\gamma = \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}.$$

By the AM-GM inequality, the inequalities $\alpha + \beta \geq 3, \beta + \gamma \geq 3, \gamma + \alpha \geq 3$ hold. Adding them together yields $2(\alpha + \beta + \gamma) \geq 9$. Using $\beta + \gamma = 3$, we obtain $\alpha \geq 3/2$. ■

§2 No square is negative ($x^2 \geq 0$)

Example 2.1 (Canada CMO 1971 P2)**.** Prove that if $x > 0, y > 0$ and $x + y = 1$, then

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9.$$

Solution 2. Note that

$$\begin{aligned}\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) &= 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy} \\ &= 1 + 1 + \frac{y}{x} + 1 + \frac{x}{y} + \frac{x+y}{xy} \\ &= 1 + 2\left(2 + \frac{x}{y} + \frac{y}{x}\right) \\ &\geq 9.\end{aligned}$$

■

Example 2.2 (USSR Olympiad 1990)**.** Prove that for arbitrary $t \in \mathbb{R}$, the inequality $t^4 - t + \frac{1}{2} > 0$ holds.

Solution 3. For any $t \in \mathbb{R}$, note that

$$\begin{aligned} t^4 - t + \frac{1}{2} &= t^4 - t^2 + \frac{1}{4} + t^2 - t + \frac{1}{4} \\ &= \left(t^2 - \frac{1}{2}\right)^2 + \left(t - \frac{1}{2}\right)^2 \geq 0, \end{aligned}$$

where equality occurs only if $t = \frac{1}{2}$ and $t^2 = \frac{1}{2}$, which is impossible. This shows that $t^4 - t + \frac{1}{2} > 0$ for any real number t. ■

Example 2.3 (India RMO 1995 P7)**.** Show that for any real number x:

$$x^2 \sin x + x \cos x + x^2 + \frac{1}{2} > 0.$$

Solution 4. When $1 + \sin x \neq 0$, we have

$$\begin{aligned} &x^2 \sin x + x \cos x + x^2 + \frac{1}{2} \\ &= (1 + \sin x)x^2 + x \cos x + \frac{1}{2} \\ &= (1 + \sin x)\left(x + \frac{\cos x}{2(1 + \sin x)}\right)^2 + \frac{1}{2} - \frac{\cos^2 x}{4(1 + \sin x)} \\ &= (1 + \sin x)\left(x + \frac{\cos x}{2(1 + \sin x)}\right)^2 + \frac{2 + 2\sin x - \cos^2 x}{4(1 + \sin x)} \\ &= (1 + \sin x)\left(x + \frac{\cos x}{2(1 + \sin x)}\right)^2 + \frac{(1 + \sin x)^2}{4} > 0. \end{aligned}$$

If $\sin x = -1$, then $x^2 \sin x + x \cos x + x^2 + \frac{1}{2}$ is equal to $\frac{1}{2}$ and hence it is positive. This completes the proof. ■

Example 2.4 (India RMO 2014d P2)**.** If x and y are positive real numbers, prove that

$$4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy.$$

Solution 5. Note that

$$\begin{aligned} 4x^4 + 4y^3 + 5x^2 + y + 1 &\geq (4x^4 + 1) + 5x^2 + (4y^3 + y) \\ &= (2x^2 - 1)^2 + 9x^2 + y(2y - 1)^2 + 4y^2 \\ &\geq 9x^2 + 4y^2 \end{aligned}$$

$$= (3x - 2y)^2 + 12xy$$
$$\geq 12xy.$$

■

Example 2.5 (India RMO 2014c P2)**.** Find all real x, y such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1).$$

Solution 6. Let x, y be reals satisfying the above inequality. Note that

$$\begin{aligned}
x^2 + 2y^2 + \frac{1}{2} - x(2y + 1) &= x^2 + 2y^2 + \frac{1}{2} - 2xy - x \\
&= 2(y^2 - xy) + x^2 - x + \frac{1}{2} \\
&= 2\left(y - \frac{x}{2}\right)^2 + \frac{x^2}{2} - x + \frac{1}{2} \\
&= 2\left(y - \frac{x}{2}\right)^2 + \frac{1}{2}(x - 1)^2 \\
&\geq 0.
\end{aligned}$$

For real x, y, the given inequality is equivalent to $x - 1 = 0, y - \frac{x}{2} = 0$, which holds if and only if (x, y) is equal to $(1, \frac{1}{2})$. ■

Example 2.6 (India RMO 2015c P7)**.** Let $x, y, z \in \mathbb{R}$, such that $x^2 + y^2 + z^2 - 2xyz = 1$. Prove that

$$(1 + x)(1 + y)(1 + z) \leq 4 + 4xyz.$$

Solution 7. Note that

$$\begin{aligned}
&4 + 4xyz - (1 + x)(1 + y)(1 + z) \\
&= 4 + 4xyz - (1 + x + y + z + x^2 + y^2 + z^2 + xyz) \\
&= 4 + 3xyz - (1 + x + y + z + x^2 + y^2 + z^2) \\
&= 4 + 3\frac{x^2 + y^2 + z^2 - 1}{2} - (1 + x + y + z + x^2 + y^2 + z^2) \\
&= \frac{1}{2}(5 + 3(x^2 + y^2 + z^2) - 2(1 + x + y + z + x^2 + y^2 + z^2)) \\
&= \frac{1}{2}(3 + x^2 + y^2 + z^2 - 2(x + y + z)) \\
&= \frac{1}{2}((x - 1)^2 + (y - 1)^2 + (z - 1)^2),
\end{aligned}$$

which is nonnegative. So the required inequality follows. ■

Example 2.7 (Putnam 1998 B1, India RMO 2015f P1)**.** Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x \in \mathbb{R}^+$.

Solution 8. Note that

$$\begin{aligned}
\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)} &= \frac{(x+1/x)^6-(x^3+1/x^3)^2}{(x+1/x)^3+(x^3+1/x^3)} \\
&= (x+1/x)^3-(x^3+1/x^3) \\
&= 3\left(x+\frac{1}{x}\right) \\
&= 6+3\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2.
\end{aligned}$$

It follows that the required minimum value is equal to 6, which is attained at $x = 1$. ■

Example 2.8 (India RMO 2018a P2)**.** Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

Solution 9. Let x be a real number satisfying the above equation. It follows that $x \neq 0$. Note that

$$\begin{aligned}
\frac{n(n+1)}{4} &= \sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} \\
&= \sum_{k=1}^{n} \frac{k}{x^k+\frac{1}{x^k}} \\
&\leq \sum_{k=1}^{n} \frac{k}{|x|^k+\frac{1}{|x|^k}} \\
&\leq \sum_{k=1}^{n} \frac{k}{2} \\
&= \frac{n(n+1)}{4}.
\end{aligned}$$

Consequently, the inequalities in the intermediate steps are equalities. This shows that x is positive and $|x| = 1$, which gives $x = 1$. Also note that the given equation holds if $x = 1$. Hence, $x = 1$ is the only real solution of the given equation. ■

§3 Manipulation

Example 3.1 (India INMO 1987 P2). Determine the largest number in the infinite sequence

$$1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \dots, n^{1/n}, \dots.$$

Solution 10. Note that $1 < 2^{1/2} < 3^{1/3}$ holds. Let us establish the following Claim.

Claim — For any integer $n \geq 3$, the inequality

$$n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$$

holds.

Proof of the Claim. Observe that the inequality is equivalent to $n > \left(1 + \frac{1}{n}\right)^n$. For any integer $n \geq 3$, note that

$$\begin{aligned}
&\left(1 + \frac{1}{n}\right)^n \\
&= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n}\frac{1}{n^n} \\
&= 1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} \\
&\qquad + \cdots + \frac{n(n-1)(n-2)\cdots 2}{(n-1)!}\frac{1}{n^{n-1}} + \frac{n(n-1)(n-2)\dots 1}{n!}\frac{1}{n^n} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \\
&\qquad \cdots + \frac{1}{(n-1)!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-2}{n}\right) \\
&\qquad\quad + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-2}{n}\right)\left(1 - \frac{n-1}{n}\right) \\
&< 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} \\
&< 2 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots + \frac{1}{\underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{(n-1)\text{-times}}} \\
&= 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \\
&< 3 \\
&\leq n
\end{aligned}$$

holds. This proves the Claim. □

It follows that the largest term in the given sequence is equal to $3^{1/3}$. ■

Remark. For any $n \geq e$, note that

$$n^{1/n} \geq e^{1/n} > 1 + \frac{1}{n}$$

holds, which shows that $n^{1/n} > (n+1)^{1/(n+1)}$.

Example 3.2 (India RMO 2000 P3)**.** Suppose $(x_1, x_2, \cdots, x_n, \cdots)$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq \cdots$, and for all n,

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all $k \geq 1$ the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \leq 3.$$

Solution 11. Note that for any integer $n \geq 1$, we have

$$\begin{aligned}
&\left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3}\right) \\
&+ \left(\frac{x_4}{4} + \frac{x_5}{5} + \frac{x_6}{6} + \frac{x_7}{7} + \frac{x_8}{8}\right) \\
&\quad + \cdots + \left(\frac{x_{n^2}}{n^2} + \frac{x_{n^2+1}}{n^2+1} + \cdots + \frac{x_{(n+1)^2-1}}{(n+1)^2-1}\right) \\
&\leq \left(\frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1}\right) + \left(\frac{x_4}{4} + \frac{x_4}{4} + \frac{x_4}{4} + \frac{x_4}{4} + \frac{x_4}{4}\right) \\
&+ \cdots + \underbrace{\left(\frac{x_{n^2}}{n^2} + \frac{x_{n^2}}{n^2} + \cdots + \frac{x_{n^2}}{n^2}\right)}_{(2n+1) \text{ terms}} \\
&= (2 \cdot 1 + 1)\frac{x_1}{1} + (2 \cdot 2 + 1)\frac{x_4}{4} + \cdots + (2n+1)\frac{x_{n^2}}{n^2} \\
&\leq (3 \cdot 1)\frac{x_1}{1} + (3 \cdot 2)\frac{x_4}{4} + \cdots + (3n)\frac{x_{n^2}}{n^2} \\
&= 3\left(\frac{x_1}{1} + \frac{x_4}{2} + \cdots + \frac{x_{n^2}}{n}\right) \\
&\leq 3.
\end{aligned}$$

Since for any $k \geq 1$, there is a positive integer m with $(m+1)^2 - 1 \geq k$, the result follows. ■

Example 3.3 (India RMO 2002 P6)**.** For any natural number $n > 1$, prove the inequality

$$\frac{1}{2} < \frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} < \frac{1}{2} + \frac{1}{2n}.$$

Solution 12. Note that

$$\begin{aligned}
&\frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} \\
&> \frac{1}{n^2+n}(1+2+3+\cdots+n) \quad (\text{using } n>1) \\
&= \frac{1}{2}.
\end{aligned}$$

Also note that

$$\begin{aligned}
&\frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} \\
&< \frac{1}{n^2+1}(1+2+3+\cdots+n) \quad (\text{using } n>1) \\
&= \frac{n^2+n}{2(n^2+1)} \\
&= \frac{1}{2} + \frac{n-1}{2(n^2+1)} \\
&< \frac{1}{2} + \frac{1}{2n}.
\end{aligned}$$

■

Example 3.4 (India RMO 2005 P3)**.** If a, b, c are three real numbers such that $|a-b| \geq |c|, |b-c| \geq |a|, |c-a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.

Solution 13. The given inequalities are equivalent to $(a-b)^2 - c^2 \geq 0, (b-c)^2 - a^2 \geq 0, (c-a)^2 - b^2 \geq 0$, which yields

$$\begin{aligned}
(a-b+c)(a-b-c) &\geq 0, \\
(b-c+a)(b-c-a) &\geq 0, \\
(c-a+b)(c-a-b) &\geq 0.
\end{aligned}$$

Multiplying them, we obtain

$$-(b+c-a)^2(c+a-b)^2(a+b-c)^2 \geq 0,$$

which shows that $(b+c-a)(c+a-b)(a+b-c)$ is equal to 0. This proves the result. ■

§4 Rearrangement inequality

Theorem 1 (Rearrangement inequality)

Let $a_1, \dots, a_n, b_1, \dots, b_n$ be real numbers satisfying $a_1 \geq a_2 \geq \cdots \geq a_n, b_1 \geq b_2 \geq \cdots \geq b_n$. Then for any permutation σ of $\{1, 2, \dots, n\}$,

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geq a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_nb_{\sigma(n)} \\ &\geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \end{aligned}$$

holds. In other words, for any two sequences $a_1, \dots, a_n$ and $b_1, \dots, b_n$ of real numbers, the sum $a_1b_1 + \cdots + a_nb_n$ is maximized (resp. minimized) when these sequences are sorted in the same (resp. opposite) order.

Example 4.1 (Canada CMO 2002 P3)**.** For positive $x, y, z \in \mathbb{R}$, prove that

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.$$

Solution 14. Since x, y, z are positive, it follows that the sequences (x^3, y^3, z^3) and $(\frac{1}{yz}, \frac{1}{zx}, \frac{1}{xy})$ are similarly ordered. By the rearrangement inequality, it follows that

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq \frac{x^3}{zx} + \frac{y^3}{xy} + \frac{z^3}{yz} = \frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y}.$$

Since the sequences (x^2, y^2, z^2) and $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$ are sorted in the opposite order, using the rearrangement inequality, we get

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} \geq \frac{x^2}{x} + \frac{y^2}{y} + \frac{z^2}{z} = x + y + z.$$

Combining the above inequalities, we obtain

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.$$

■

Example 4.2. For positive reals a, b, c, show that $a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3$.

Solution 15. Since a, b, c are positive reals, it follows that the sequences a^4, b^4, c^4 and a^3, b^3, c^3 are sorted in the same order. By the rearrangement inequality, we obtain

$$a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3.$$

■

Example 4.3 (India RMO 2012a P3, India RMO 2012b P3, India RMO 2012c P3, India RMO 2012d P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \leq 1$.

Solution 16. For any positive a, b, the sequences $(a^a, b^a), (a^b, b^b)$ are sorted the same way. Applying the rearrangement inequality, we obtian

$$a^a b^b + a^b b^a \leq a^a a^b + b^a b^b = a^{a+b} + b^{a+b} = a + b = 1.$$

■

Example 4.4 (India RMO 2014b P2)**.** Let x, y, z be positive real numbers. Prove that

$$\frac{y^2+z^2}{x} + \frac{z^2+x^2}{y} + \frac{x^2+y^2}{z} \geq 2(x+y+z).$$

Solution 17. Note that

$$\frac{y^2+z^2}{x} + \frac{z^2+x^2}{y} + \frac{x^2+y^2}{z} = \left(\frac{y^2}{x} + \frac{z^2}{y} + \frac{x^2}{z}\right) + \left(\frac{z^2}{x} + \frac{x^2}{y} + \frac{y^2}{z}\right).$$

holds. Since (x^2, y^2, z^2) and $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$ are sorted oppositely, by the rearrangement inequality, we obtain

$$\left(\frac{y^2}{x} + \frac{z^2}{y} + \frac{x^2}{z}\right) \geq x+y+z, \quad \left(\frac{z^2}{x} + \frac{x^2}{y} + \frac{y^2}{z}\right) \geq x+y+z.$$

Adding these inequalities, the required inequality follows. ■

> **Remark.** Note that the last two inequalities also follows from the Cauchy–Schwarz inequality.

Example 4.5 (India INMO 2001 P3)**.** If a, b, c are positive real numbers such that $abc = 1$, Prove that

$$a^{b+c} b^{c+a} c^{a+b} \leq 1.$$

Solution 18. Put

$$\alpha = a^{b+c} b^{c+a} c^{a+b},$$
$$\beta = b^{b+c} c^{c+a} a^{a+b},$$
$$\gamma = c^{b+c} a^{c+a} b^{a+b}.$$

Note that $\alpha\beta\gamma = 1$, and by the rearrangement inequality, it follows that

$$\alpha \leq \beta, \alpha \leq \gamma$$

hold. This gives $\alpha^3 \leq 1$, which yields $\alpha \leq 1$. ■

> **Remark.** The above proof is similar to be the proof of Nesbitt's inequality as in [**Hun08**].

Example 4.6 (India RMO 2017a P6)**.** Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}$$

Solution 19. Note that both sides of the above inequality remain invariant under cyclic permutations of x, y, z. Therefore, without loss of generality, we assume that $x \geq y$ and $x \geq z$. Note that

$$\begin{aligned}\frac{x-1}{y-1} - \frac{x+1}{y+1} &= \frac{(x-1)(y+1) - (x+1)(y-1)}{y^2-1} \\ &= \frac{2(x-y)}{y^2-1}.\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}\frac{y-1}{z-1} - \frac{y+1}{z+1} &= \frac{2(y-z)}{z^2-1}, \\ \frac{z-1}{x-1} - \frac{z+1}{x+1} &= \frac{2(z-x)}{x^2-1}.\end{aligned}$$

It suffices to prove that

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} \geq 0,$$

which is equivalent to

$$\frac{x}{y^2-1} + \frac{y}{z^2-1} + \frac{z}{x^2-1} \geq \frac{x}{x^2-1} + \frac{y}{y^2-1} + \frac{z}{z^2-1},$$

which follows from the rearrangement inequality since x, y, z and $\frac{1}{x^2-1}, \frac{1}{y^2-1}, \frac{1}{z^2-1}$ are sorted in the opposite order. ■

§5 Cauchy–Schwarz inequality

Example 5.1. Show that if the sum of positive numbers a, b, c is equal to 1, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

Solution 20. Applying the Cauchy–Schwarz inequality, we obtain

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3^2.$$

Using $a+b+c=1$, the required inequality follows. ■

Example 5.2 (India RMO 2013d P3)**.** Given real numbers $a, b, c, d, e > 1$. Prove that

$$\frac{a^2}{c-1}+\frac{b^2}{d-1}+\frac{c^2}{e-1}+\frac{d^2}{a-1}+\frac{e^2}{b-1} \geq 20.$$

Solution 21. Since $a-1, b-1, c-1, d-1, e-1$ are all positive, by applying the Cauchy-Schwarz inequality, we get

$$\frac{a^2}{c-1}+\frac{b^2}{d-1}+\frac{c^2}{e-1}+\frac{d^2}{a-1}+\frac{e^2}{b-1} \geq \frac{(a+b+c+d+e)^2}{a+b+c+d+e-5}.$$

It suffices to show that

$$\frac{(a+b+c+d+e)^2}{a+b+c+d+e-5} \geq 20$$

holds. Since $a+b+c+d+e-5$ is positive, it is enough to prove that

$$(a+b+c+d+e)^2 - 20(a+b+c+d+e) + 100 \geq 0,$$

which holds

$$(a+b+c+d+e)^2 - 20(a+b+c+d+e) + 100 = (a+b+c+d+e-10)^2.$$

This proves the result. ■

Example 5.3 (India RMO 2014a P6)**.** Let $x_1, x_2, x_3, \dots, x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq 1. \tag{1}$$

Solution 22. For any positive reals $x_1, \dots, x_{2014}$ satisfying $\sum_{j=1}^{2014} x_j = 1$, using the Cauchy–Schwarz inequality, we obtain

$$\sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq \frac{(x_1+\cdots+x_{2014})^2}{2014-(x_1+\cdots+x_{2014})} = \frac{1}{2013},$$

which gives

$$2013 \sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq 1.$$

This shows that the inequality (1) holds for $K = 2013$.

For $x_1 = x_2 = \cdots = x_{2014} = \frac{1}{2014}$, note that

$$\sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} = \frac{1}{2013}$$

holds, which shows that for any K satisfying the inequality (1), the bound $K \geq 2013$ holds.

This proves that the smallest constant K satisfying the inequality (1) is equal to 2013. ■

Example 5.4 (India RMO 2016d P2)**.** Let a, b, c be positive real numbers such that

$$\frac{ab}{1 + bc} + \frac{bc}{1 + ca} + \frac{ca}{1 + ab} = 1.$$

Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}.$$

We refer to Example 9.1 and Example 9.1.

Solution 23. Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} 1 &= \frac{ab}{1 + bc} + \frac{bc}{1 + ca} + \frac{ca}{1 + ab} \\ &\geq \frac{(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2}{3 + ab + bc + ca}, \end{aligned}$$

which yields

$$3 \geq 2\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Using the AM-GM inequality, we get

$$2\sqrt{abc}\sqrt[3]{\sqrt{abc}} \leq 1,$$

which gives $abc \leq \frac{1}{2\sqrt{2}}$. Note that

$$\begin{aligned} \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} &\geq \frac{3}{abc} \\ &\geq 6\sqrt{2}. \end{aligned}$$

■

Example 5.5 (India RMO 2023b P5)**.** Let $n > k > 1$ be positive integers. Determine all positive real numbers $a_1, a_2, \ldots, a_n$ which satisfy

$$\sum_{i=1}^{n} \sqrt{\frac{k a_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n.$$

Solution 24. Let $a_1, \ldots, a_n$ be positive reals satisfying the above condition. Note that

$$\begin{aligned}
&\sum_{i=1}^{n} \sqrt{\frac{k a_i^k}{(k-1)a_i^k + 1}} \\
&\leq \sum_{i=1}^{n} \sqrt{\frac{a_i^k}{\sqrt[k]{a_i^{k(k-1)}}}} \quad \text{(by the AM-GM inequality)} \\
&= \sum_{i=1}^{n} \sqrt{a_i} \\
&\leq \left(n \sum_{i=1}^{n} a_i \right)^{1/2} \quad \text{(by the Cauchy–Schwarz inequality)} \\
&= \sum_{i=1}^{n} \sqrt{\frac{k a_i^k}{(k-1)a_i^k + 1}}
\end{aligned}$$

hold, and hence, the intermediate inequalities are equalities. This implies that $a_1, a_2, \ldots, a_n$ are equal. Using the given condition, it follows that

$$a_1 = a_2 = \cdots = a_n = 1.$$

Also note that the given conditions are satisfied if $a_1 = a_2 = \cdots = a_n = 1$ holds. This proves that the positive real numbers satisfying the given condition are precisely

$$a_1 = a_2 = \cdots = a_n = 1.$$

■

§6 QM-AM-GM-HM inequality

Example 6.1. Prove that if $m > 0$, then

$$m + \frac{4}{m^2} \geq 3.$$

Solution 25. Applying the AM-GM inequality, it follows that

$$m + \frac{4}{m^2} = \frac{m}{2} + \frac{m}{2} + \frac{4}{m^2} \geq 3.$$

■

Example 6.2 (India INMO 1988 P4)**.** If $a, b > 0$ with $a + b = 1$, then show that $(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 \geq \frac{25}{2}$.

Solution 26. Note that

$$\begin{aligned}\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 &= a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}+4\\ &= a^2+b^2+1+\frac{b^2}{a^2}+2\frac{b}{a}+1+\frac{a^2}{b^2}+2\frac{a}{b}+4\\ &= a^2+b^2+\left(\frac{a}{b}-\frac{b}{a}\right)^2+\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2+12\\ &\geq \frac{(a+b)^2}{2}+12\\ &= \frac{25}{2}.\end{aligned}$$

■

Example 6.3 (India RMO 1991 P2)**.** If a, b, c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq 4.$$

Solution 27. Note that

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} &\geq 2\sqrt{\frac{a}{c}}+2\sqrt{\frac{c}{a}}\\ &\geq 4.\end{aligned}$$

■

Example 6.4 (All-Russian MO 1991 Grade 11 First Day P3, India RMO 1994 P8)**.** If a, b and c are positive real numbers such that $a+b+c=1$, prove that

$$(1+a)(1+b)(1+c)\geq 8(1-a)(1-b)(1-c).$$

Solution 28. The given inequality is equivalent to

$$(b+c+2a)(c+a+2b)(a+b+2c)\geq 8(b+c)(c+a)(a+b).$$

Put $x=b+c, y=c+a, z=a+b$. Note that the above inequality can be rewritten as

$$(y+z)(z+x)(x+y)\geq 8xyz,$$

which follows from the AM-GM inequality. ■

Example 6.5 (India RMO 1993 P6). If a, b, c, d are four positive real numbers such that $abcd = 1$, prove that

$$(1+a)(1+b)(1+c)(1+d) \geq 16.$$

Solution 29. It follows by applying the AM-GM inequality to the factors and using $abcd = 1$. ■

Example 6.6. Show that if $a, b, c \geq 0$, then

$$ab(a+b) + bc(b+c) + ca(c+a) \geq 6abc.$$

Example 6.7. [**PK74**, Problem 62.2] Each of the four numbers a, b, c, and d is positive and less than one. Show that not all four products

$$4a(1-b), 4b(1-c), 4c(1-d), 4d(1-a)$$

are greater than one.

Solution 30. Note that their product is equal to

$$\prod_{x \in \{a,b,c,d\}} 4x(1-x) = \prod_{x \in \{a,b,c,d\}} (1 - (2x-1)^2)$$

which is at most 1. This shows that not all the above four products are greater than one. ■

Example 6.8 (India RMO 2003 P3). Let a, b, c be three positive real numbers such that $a+b+c = 1$. Prove that among the three numbers $a - ab, b - bc, c - ca$, there is one which is at most $\frac{1}{4}$ and there is one which is at least $\frac{2}{9}$.

Solution 31. Note that

$$(a - ab)(b - bc)(c - ca) = a(1-a)b(1-b)c(1-c) \leq \frac{1}{4^3}$$

holds. So among $a - ab, b - bc, c - ca$ there is one which is at most $\frac{1}{4}$.

Also note that

$$\begin{aligned}(a - ab) + (b - bc) + (c - ca) &= 1 - (ab + bc + ca)\\ &\geq 1 - \frac{1}{3}(a+b+c)^2\\ &\geq \frac{2}{3}\end{aligned}$$

holds. This shows that one of $a - ab, b - bc, c - ca$ is at least $\frac{2}{9}$. ■

Example 6.9 (India RMO 2005 P7). Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let

$$\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

Solution 32. We need to show that the discriminant of $x^2 + x + 4\lambda$ is nonnegative, that is, $\lambda \leq \frac{1}{16}$ holds. On the contrary, let us assume that $\lambda > \frac{1}{16}$. This gives

$$a^3 + a^2bc > \frac{1}{16}, b^3 + ab^2c > \frac{1}{16}, c^3 + abc^2 > \frac{1}{16}.$$

Note that

$$a^3 + a^2bc = a^2(a + bc) = a^2(1 - b - c + bc) = a^2(1 - b)(1 - c) > \frac{1}{16}$$

holds. Similarly, it follows that

$$b^2(1 - c)(1 - a) > \frac{1}{16}, c^2(1 - a)(1 - b) > \frac{1}{16}.$$

This implies that

$$(abc(1 - a)(1 - b)(1 - c))^2 > \frac{1}{16^3},$$

which is impossible since

$$a(1 - a) \leq \frac{1}{4}, b(1 - b) \leq \frac{1}{4}, c(1 - c) \leq \frac{1}{4}.$$

Consequently, we obtain $\lambda \leq \frac{1}{16}$, and hence the roots of the equation $x^2 + x + 4\lambda = 0$ are real. ■

Example 6.10. If a, b, c are positive reals, then show that

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq (a + b)(b + c)(c + a).$$

Solution 33. Note that

$$(a^2 + 1)(b^2 + 1) - (a + b)^2 = (ab - 1)^2 \geq 0$$

holds. Similarly, it follows that

$$(b^2 + 1)(c^2 + 1) \geq (b + c)^2, (c^2 + 1)(a^2 + 1) \geq (c + a).$$

Multiplying the above inequalities and using that a, b, c are nonnegative, the required inequality follows. ■

Example 6.11 (India RMO 2006 P3)**.** If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq 3.$$

Solution 34. Note that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \geq 3,$$

where the first inequality follows from AM-GM inequality and the second inequality follows from Nesbitt's inequality Example 1.1. ■

Example 6.12 (India RMO 2007 P6)**.** Prove that

(a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$,

(b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$,

(c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Solution 35. Note that

$$\sqrt{5} > 2, \sqrt[3]{5} > \frac{8}{5}, \sqrt[4]{5} > \frac{7}{5}$$

holds, which gives the first inequality. Using $3 > \sqrt{8}, 3 > \sqrt[4]{8}$, we obtain the second inequality. Note that

$$\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} < \sqrt{n} + \sqrt{n} + \sqrt{n} = 3\sqrt{n}$$

holds for any integer $n > 1$. Since $n \geq 3\sqrt{n}$ holds for any $n \geq 9$, the third inequality follows. ■

> **Remark.** One may also use the AM-GM inequality to obtain the first inequality.

Example 6.13 (India RMO 2008 P3)**.** Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1$ are positive real numbers. Prove that

(i) $0 < 3ab \leq 1$,

(ii) $b \geq \sqrt{3}$.

Solution 36. Let α, β, γ denote the roots of the polynomial $ax^3 - x^2 + bx - 1$. Note that

$$\alpha + \beta + \gamma = \frac{1}{a},$$

$$\begin{aligned}\alpha\beta + \beta\gamma + \gamma\alpha &= \frac{b}{a},\\ \alpha\beta\gamma &= \frac{1}{a}\end{aligned}$$

holds. Since α, β, γ are positive, it follows that a, b are positive. Using

$$(\alpha + \beta + \gamma)^2 \geq 3(\alpha\beta + \beta\gamma + \gamma\alpha),$$

we obtain $\frac{1}{a^2} \geq \frac{3b}{a}$. Since a is positive, it follows that $3ab \leq 1$. Note that

$$\begin{aligned}(\alpha\beta + \beta\gamma + \gamma\alpha)^2 &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)\\ &\geq \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2}{3} + 2\alpha\beta\gamma(\alpha + \beta + \gamma)\end{aligned}$$

holds, we get $\frac{b^2}{a^2} \geq \frac{b^2}{3a^2} + 2\frac{1}{a^2}$, which gives $b^2 \geq 3$. Since b is positive, we conclude $b \geq \sqrt{3}$. ■

Example 6.14 (India RMO 2011b P6)**.** Find the largest real constant λ such that

$$\frac{\lambda abc}{a + b + c} \leq (a + b)^2 + (a + b + 4c)^2 \tag{2}$$

for all positive real numbers a, b, c.

Solution 37. Let λ be a nonnegative real number. Then the above inequality holds for all $a, b, c > 0$ if and only if

$$\lambda x^2 c \leq (2x + c)(4x^2 + 4(x + 2c)^2)$$

holds for all $x, c > 0$. Noting that both the sides are homogeneous of degree three in x and y, it follows that the above inequality holds if and only if

$$\lambda x^2 \leq (2x + 1)(4x^2 + 4(x + 2)^2)$$

holds for all $x > 0$. Observe that

$$\begin{aligned}(2x + 1)(4x^2 + 4(x + 2)^2) &= 8(2x + 1)(x^2 + 2x + 2)\\ &= 8(2x^3 + 5x^2 + 6x + 2).\end{aligned}$$

This shows that for $x > 0$, the above inequality is equaivalent to

$$\frac{1}{2}\left(\frac{\lambda}{8} - 5\right) \leq x + \frac{3}{x} + \frac{1}{x^2}.$$

Let $a, b > 0$ be such that

$$x + \frac{3}{x} + \frac{1}{x^2} = ax + \frac{3}{x} + 2bx + \frac{1}{x^2}$$

holds for any $x > 0$, and $(3/a)^3 = (1/b)^2$ holds, or equivalently, a, b satisfy $a + 2b = 1$ and $(3/a)^3 = (1/b)^2$. Note that this holds for $a = 3/4, b = 1/8$. Observe that

$$x + \frac{3}{x} + \frac{1}{x^2} = \frac{3}{4}x + \frac{3}{x} + \frac{1}{8}x + \frac{1}{8}x + \frac{1}{x^2} \geq 3 + \frac{3}{4} = \frac{15}{4},$$

where equality holds if and only if $x = 2$. This proves that the largest real constant λ satisfying the given inequality for all $a, b, c > 0$ satisfies

$$\frac{1}{2}\left(\frac{\lambda}{8} - 5\right) = \frac{15}{4},$$

or equivalently, $\lambda = 100$ holds. ■

Example 6.15 (India RMO 2012e P4). Let a, b, c be positive real numbers such that $abc(a + b + c) = 3$. Prove that we have

$$(a + b)(b + c)(c + a) \geq 8.$$

Also determine the case of equality.

Solution 38. Note that

$$\begin{aligned}(a + b)(b + c)(c + a) &= (a + b + c)(ab + bc + ca) - abc \\ &= abc(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - abc \\ &= 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - abc \\ &\geq 9\frac{1}{\sqrt[3]{abc}} - abc.\end{aligned}$$

Also note that

$$\begin{aligned}3 &= abc(a + b + c) \\ &\geq 3(abc)^{4/3},\end{aligned}$$

which gives $abc \leq 1$. Using the above inequalities, we obtain

$$\begin{aligned}(a + b)(b + c)(c + a) &\geq 9 - 1 \\ &= 8.\end{aligned}$$

This proves that

$$(a + b)(b + c)(c + a) \geq 8,$$

where equality holds if and only if all the prior inequalities are equalities, or equivalently, $a = b = c$. Using $abc(a+b+c) = 3$, it follows that a, b, c are equal if and only if they are equal to 1. This shows that $(a + b)(b + c)(c + a) = 8$ if and only if a, b, c are equal to 1. ■

Example 6.16 (India RMO 2012f P8)**.** Let x, y, z be positive real numbers such that $2(xy + yz + zx) = xyz$. Prove that

$$\frac{1}{(x-2)(y-2)(z-2)} + \frac{8}{(x+2)(y+2)(z+2)} \leq \frac{1}{32}.$$

If a, b, c are positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, then show that

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \leq \frac{1}{4}.$$

Solution 39. Put

$$x = 2a, y = 2b, z = 2c.$$

Note that a, b, c are positive real numbers and they satisfy $ab + bc + ca = abc$. Observe that

$$\begin{aligned}(a-1)(b-1)(c-1) &= abc - ab - bc - ca + a + b + c - 1 \\ &= a + b + c - 1, \\ (a+1)(b+1)(c+1) &= abc + ab + bc + ca + a + b + c + 1 \\ &= 2abc + a + b + c + 1.\end{aligned}$$

Using the AM-GM-HM inequality, we obtain

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}},$$

which yields $a + b + c \geq 9$ and $abc \geq 3^3$. This implies that

$$\begin{aligned}&\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \\ &\leq \frac{1}{9-1} + \frac{8}{2 \cdot 27 + 9 + 1} \\ &= \frac{1}{8} + \frac{8}{64} \\ &= \frac{1}{4}.\end{aligned}$$

■

Example 6.17 (ELMO 2013 P2, proposed by Evan Chen)**.** Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Solution 40. Using the weighted AM-GM inequality, we obtain

$$1 = \sum_{\text{cyc}} \frac{a}{a+b+c} \cdot a^{-\frac{6}{7}} \geq (a^a b^b c^c)^{-\frac{6/7}{a+b+c}},$$

which yields $a^a b^b c^c \geq 1$. ■

Example 6.18 (India RMO 2014e P5)**.** Let a, b, c be positive real numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1. \tag{3}$$

Prove that

$$(1+a^2)(1+b^2)(1+c^2) \geq 125.$$

When does the equality hold?

Solution 41. The given inequality implies

$$\frac{a}{1+a} \geq \frac{1}{1+b} + \frac{1}{1+c} \geq \frac{2}{\sqrt{(1+b)(1+c)}},$$

where the second inequality holds if and only if $b = c$. Similar lower bounds for $\frac{b}{1+b}, \frac{c}{1+c}$ can be obtained, and multiplying them yields $abc \geq 8$, where equality holds if and only if $a = b = c = 2$. Note that

$$1 + a^2 = 1 + \frac{a^2}{4} + \frac{a^2}{4} + \frac{a^2}{4} + \frac{a^2}{4} \geq 5\left(\frac{a^8}{2^8}\right)^{\frac{1}{5}}$$

holds, where equality occurs if and only if $a = 2$. We can find similar lower bounds for $1 + b^2, 1 + c^2$. Multiplying them, we get

$$(1+a^2)(1+b^2)(1+c^2) \geq 5^3\left(\frac{a^8}{2^8}\frac{b^8}{2^8}\frac{c^8}{2^8}\right)^{\frac{1}{5}} = 5^3\left(\frac{abc}{8}\right)^{\frac{8}{5}} \geq 125,$$

where equality occurs if and only if $a = b = c = 2$. ■

Example 6.19 (India RMO 2016c P2)**.** Let a, b, c be three distinct positive real numbers such that $abc = 1$. Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

Solution 42. Note that

$$\begin{aligned}
&\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \\
&= -\frac{a^3(b-c) + b^3(c-a) + c^3(a-b)}{(a-b)(b-c)(c-a)} \\
&= -\frac{a^3(b-c) + b^3(c-a) - c^3(b-c) - c^3(c-a)}{(a-b)(b-c)(c-a)} \\
&= -\frac{(b^3-c^3)(c-a) + (a^3-c^3)(b-c)}{(a-b)(b-c)(c-a)}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{b^2+bc+c^2-c^2-a^2-ca}{a-b} \\
&= -\frac{b^2+bc-a^2-ca}{a-b} \\
&= a+b+c \\
&\geq 3\sqrt[3]{abc} \\
&= 3.
\end{aligned}
$$

■

Remark. The above argument leads to following somewhat simpler solution. Observing that

$$
\begin{aligned}
\frac{1}{(c-a)(c-b)} &= \frac{(c-a)-(c-b)}{(b-a)(c-a)(c-b)} \\
&= \frac{1}{(b-a)(c-b)} - \frac{1}{(b-a)(c-a)},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \\
&= \frac{a^3-c^3}{(a-b)(a-c)} + \frac{b^3-c^3}{(b-c)(b-a)} \\
&= \frac{a^2+ac+c^2}{a-b} - \frac{b^2+bc+c^2}{a-b} \\
&= a+b+c \\
&\geq 3.
\end{aligned}
$$

Example 6.20 (India RMO 2016e P4)**.** Let a, b, c be positive real numbers such that $a+b+c=3$. Determine, with certainty, the largest possible value of the expression

$$\frac{a}{a^3+b^2+c} + \frac{b}{b^3+c^2+a} + \frac{c}{c^3+a^2+b}.$$

Solution 43. Note that

$$
\begin{aligned}
&\frac{a}{a^3+b^2+c} + \frac{b}{b^3+c^2+a} + \frac{c}{c^3+a^2+b} \\
&= \frac{1}{a^2+\frac{b^2}{a}+\frac{c}{a}} + \frac{1}{b^2+\frac{c^2}{b}+\frac{a}{b}} + \frac{1}{c^2+\frac{a^2}{c}+\frac{b}{c}} \\
&\leq \frac{1+a+ca}{(a+b+c)^2} + \frac{1+b+ab}{(a+b+c)^2} + \frac{1+c+bc}{(a+b+c)^2}
\end{aligned}
$$

$$\begin{aligned}&\text{(by the Cauchy–Schwarz inequality)}\\ &= \frac{3+a+b+c+ab+bc+ca}{(a+b+c)^2}\\ &\le \frac{6+\frac{(a+b+c)^2}{3}}{9}\\ &= 1.\end{aligned}$$

Also note that if $a=b=c=1$, then $a+b+c=3$ and

$$\frac{a}{a^3+b^2+c}+\frac{b}{b^3+c^2+a}+\frac{c}{c^3+a^2+b}=1.$$

This shows that the largest possible value of the given expression is equal to 1. ■

Example 6.21 (India RMO 2016f P5)**.** Let x, y, z be non-negative real numbers such that $xyz = 1$. Prove that

$$(x^3+2y)(y^3+2z)(z^3+2x) \ge 27.$$

Solution 44. Applying the AM-GM inequality, we obtain

$$\begin{aligned}&(x^3+2y)(y^3+2z)(z^3+2x)\\ &\ge (3\sqrt[3]{x^3y^2})(3\sqrt[3]{y^3z^2})(3\sqrt[3]{z^3x^2})\\ &= 27.\end{aligned}$$

■

Example 6.22 (India RMO 2019a P3)**.** Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+a^3+c^3}+\frac{c}{c^2+a^3+b^3} \le \frac{1}{5abc}.$$

> **Walkthrough** — **Homogenize** the denominator and apply the AM-GM inequality.

Solution 45. Note that

$$\begin{aligned}\frac{a}{a^2+b^3+c^3} &= \frac{a}{a^2(a+b+c)+b^3+c^3}\\ &\le \frac{a}{5\sqrt[5]{a^6\cdot abc\cdot b^3\cdot c^3}}\\ &= \frac{1}{5\sqrt[5]{a^2b^4c^4}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{5abc}\left(a^3bc\right)^{1/5} \\
&\leq \frac{3a+b+c}{25abc}
\end{aligned}
$$

holds. Similarly, it follows that

$$
\begin{aligned}
\frac{b}{b^2+a^3+c^3} &\leq \frac{3b+a+c}{25abc}, \\
\frac{c}{c^2+a^3+b^3} &\leq \frac{3c+a+b}{25abc}.
\end{aligned}
$$

Adding the above inequalities, we obtain

$$
\begin{aligned}
&\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+a^3+c^3} + \frac{c}{c^2+a^3+b^3} \\
&\leq \frac{3a+b+c}{25abc} + \frac{3b+a+c}{25abc} + \frac{3c+a+b}{25abc} \\
&= \frac{a+b+c}{5abc} \\
&= \frac{1}{5abc}.
\end{aligned}
$$

■

§7 Bunching terms

Example 7.1. Show that

$$
\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^m} \geq \frac{m}{2} \quad \text{for } m \geq 1.
$$

Use it to show that $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$ becomes arbitrarily large as n increases indefinitely.

Walkthrough — Observe that

$$
\begin{aligned}
&\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^m} \\
&= \frac{1}{2} \\
&\quad + \left(\frac{1}{3}+\frac{1}{4}\right) \\
&\quad + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) \\
&\quad + \ldots
\end{aligned}
$$

$$+\left(\frac{1}{2^{m-1}}+\frac{1}{2^{m-1}+1}+\cdots+\frac{1}{2^m}\right).$$

Show that the sum in each parenthesis is at most $\frac{1}{2}$.

Example 7.2 (India RMO 1992 P6)**.** Show that

$$1<\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{3001}<\frac{4}{3}$$

Solution 46. To obtain the lower bound, note that

$$\begin{aligned}
&\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{3001}\\
&>\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{3000}\\
&=\left(\frac{1}{1001}+\cdots+\frac{1}{1250}\right)+\left(\frac{1}{1251}+\cdots+\frac{1}{1500}\right)+\left(\frac{1}{1501}+\cdots+\frac{1}{1750}\right)\\
&+\left(\frac{1}{1751}+\cdots+\frac{1}{2000}\right)+\left(\frac{1}{2001}+\cdots+\frac{1}{2250}\right)+\left(\frac{1}{2251}+\cdots+\frac{1}{2500}\right)\\
&+\left(\frac{1}{2501}+\cdots+\frac{1}{2750}\right)+\left(\frac{1}{2751}+\cdots+\frac{1}{3000}\right)\\
&>\frac{250}{1250}+\frac{250}{1500}+\frac{250}{1750}+\frac{250}{2000}+\frac{250}{2250}+\frac{250}{2500}+\frac{250}{2750}+\frac{250}{3000}\\
&>\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12}=17\left(\frac{1}{60}+\frac{1}{66}+\frac{1}{70}+\frac{1}{72}\right)>1.
\end{aligned}$$

To get the upper bound, note that

$$\begin{aligned}
&\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{3001}\\
&<\frac{1}{1000}+\frac{1}{1001}+\cdots+\frac{1}{3000}\\
&<\left(\frac{1}{1000}+\frac{1}{1001}+\cdots+\frac{1}{1499}\right)+\left(\frac{1}{1500}+\frac{1}{1501}+\cdots+\frac{1}{1999}\right)\\
&+\left(\frac{1}{2000}+\frac{1}{2001}+\cdots+\frac{1}{2499}\right)+\left(\frac{1}{2500}+\frac{1}{2501}+\cdots+\frac{1}{2999}\right)+\frac{1}{3000}\\
&<\frac{500}{1500}+\frac{500}{2000}+\frac{500}{2500}+\frac{500}{3000}+\frac{1}{3000}\\
&=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{3000}\\
&=\frac{57}{60}+\frac{1}{3000}\\
&<1
\end{aligned}$$

$$< \frac{4}{3}.$$

■

Example 7.3 (Canada CMO 1998 P3, India RMO 1998 P3)**.** Prove the following inequality for every natural number $n \geq 2$:

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right).$$

Solution 47. Let us establish the following Claim.

Claim — For any $1 \leq k \leq n$, the inequality

$$\frac{1}{n+1}\left(\frac{1}{2k-1}+\frac{1}{2n-2k+1}\right) \geq \frac{1}{n}\left(\frac{1}{2k}+\frac{1}{2n-2k+2}\right)$$

holds. Moreover, this inequality is strict if $k = 1$ and $n \geq 2$.

Proof of the Claim. Clearing the denominators, it follows that the above inequality is equivalent to

$$4n^2k(n-k+1) \geq (n+1)^2(2k-1)(2n-2k+1).$$

Note that

$$\begin{aligned}
&4n^2k(n-k+1) - (n+1)^2(2k-1)(2n-2k+1) \\
&= n^2(4nk - 4k^2 + 4k - (4nk - 4k^2 + 4k - 2n - 1)) \\
&\quad - (2n+1)(2k-1)(2n-2k+1) \\
&= n^2(2n+1) - (2n+1)(2k-1)(2n-2k+1) \\
&= (2n+1)(n^2 - (4nk - 4k^2 + 4k - 2n - 1)) \\
&= (2n+1)((n-2k)^2 + 2(n-2k) + 1) \\
&= (2n+1)(n-2k+1)^2,
\end{aligned}$$

which is nonnegative for any $1 \leq k \leq n$. This proves the inequality. Moreover, if $k = 1$ and $n \geq 2$, then

$$4n^2k(n-k+1) - (n+1)^2(2k-1)(2n-2k+1) > 0,$$

which proves the second statement. □

The Claim implies the given inequality. ■

§8 Look at the difference

Example 8.1. Show that if $a > b > 0$, then

$$a + \frac{1}{(a-b)b} \geq 3.$$

Solution 48. Applying the AM-GM inequality, we obtain

$$a + \frac{1}{(a-b)b} = (a-b) + b + \frac{1}{(a-b)b} \geq 3.$$

■

Example 8.2. Prove that if $a < b < c < d$, then $(a+b+c+d)^2 > 8(ac+bd)$.

Solution 49. Let x, y, z denote the real numbers satisfying

$$\begin{aligned} b &= a + x, \\ c &= b + y, \\ d &= c + z. \end{aligned}$$

Note that

$$\begin{aligned} a+b+c+d &= a+b+2c+(d-c) \\ &= a+3b+2(c-b)+(d-c) \\ &= 4a+3(b-a)+2(c-b)+(d-c) \\ &= 4a+3x+2y+z, \\ ac+bd &= ac+(a+x)(c+z) \\ &= 2ac+cx+az+zx \\ &= 2a(b+y)+x(b+y)+az+zx \\ &= 2a(a+x+y)+x(a+x+y)+az+zx \\ &= 2a^2+2ax+2ay+ax+x^2+xy+az+zx \\ &= 2a^2+3ax+2ay+zx+x^2+xy+az \end{aligned}$$

hold. This shows that

$$\begin{aligned} &(a+b+c+d)^2 - 8(ac+bd) \\ &= (4a+3x+2y+z)^2 - 8(2a^2+3ax+2ay+zx+x^2+xy+az) \\ &= 9x^2+4y^2+z^2+24ax+16ay+8az+12xy+4yz+6zx \\ &\quad -24ax-16ay-8zx-8x^2-8xy-8az \\ &= x^2+4y^2+z^2+4xy+4yz-2zx \end{aligned}$$

$$= (x-z)^2 + 4y(x+y+z)$$
$$> 0.$$

■

Example 8.3 (India RMO 2004 P7)**.** Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that

1. $y < x < 1$ and
2. $x^2 + y^2 < 1$.

Solution 50. Note that $x - y \geq x^3 + y^3 > 0$ holds, which shows that $x > y$. Also note that $x - x^3 \geq y^3 + y > 0$ holds, which implies that $x(1-x^2) > 0$. Since x is positive, we obtain $x < 1$. This gives $0 < y < x < 1$.

Write $x = y + t$ with $t > 0$. The inequality $y^3 + y \leq x - x^3$ yields

$$2y^3 + 3y^2t + 3yt + t^3 \leq t.$$

Note that the inequality $x^2 + y^2 < 1$ is equivalent to $2y^2 + 2yt + t^2 < 1$. Since $y + t$ is positive, it is equivalent to $(y+t)(2y^2 + 2yt + t^2) < y + t$. Observe that

$$\begin{aligned}(y+t)(2y^2+2yt+t^2) &= 2y^3 + 4y^2t + 3yt^2 + t^3 \\ &\leq y^2t + 3yt^2 + t - 3yt \\ &= y^2t - 3yt(1-t) + t \\ &< y + t \quad (\text{using } 0 < y, t < 1).\end{aligned}$$

This completes the proof. ■

Example 8.4 (India RMO 2017b P5)**.** If $a, b, c, d \in \mathbb{R}$ such that $a > b > c > d > 0$ and $a + d = b + c$; then prove that

$$\frac{(a+b)-(c+d)}{\sqrt{2}} > \sqrt{a^2+b^2} - \sqrt{c^2+d^2}.$$

Solution 51. Put $m = a + d$. Using $a + d = b + c$ and $a > b > c > d > 0$, it follows that for some real numbers $0 < y < x < m$,

$$a = m + x, b = m + y, c = m - y, d = m - x$$

hold. Note that the given inequality reduces to

$$\sqrt{2}(x+y) > \sqrt{(m+x)^2 + (m+y)^2} - \sqrt{(m-x)^2 + (m-y)^2}.$$

Observe that

$$\sqrt{(m+x)^2 + (m+y)^2} - \sqrt{(m-x)^2 + (m-y)^2}$$

$$\begin{aligned}&= \frac{(m+x)^2+(m+y)^2-(m-x)^2-(m-y)^2}{\sqrt{(m+x)^2+(m+y)^2}+\sqrt{(m-x)^2+(m-y)^2}}\\&= \frac{4m(x+y)}{\sqrt{(m+x)^2+(m+y)^2}+\sqrt{(m-x)^2+(m-y)^2}}.\end{aligned}$$

Hence, it suffices to prove that

$$\sqrt{(m+x)^2+(m+y)^2}+\sqrt{(m-x)^2+(m-y)^2} > 2\sqrt{2}m.$$

Note that

$$\begin{aligned}&\sqrt{(m+x)^2+(m+y)^2}+\sqrt{(m-x)^2+(m-y)^2}\\&> \frac{m+x+m+y}{\sqrt{2}}+\frac{m-x+m-y}{\sqrt{2}}\\&= 2\sqrt{2}m\end{aligned}$$

hold. This completes the proof. ■

Example 8.5 (India RMO 2024b P4)**.** Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2+a_2^2+a_3^2+a_4^2 = 1$. Show that there exist i, j with $1 \leq i < j \leq 4$, such that $(a_i - a_j)^2 \leq \frac{1}{5}$.

Solution 52. Reording a_1, a_2, a_3, a_4 if necessary, we assume that $a_1 \leq a_2 \leq a_3 \leq a_4$ holds. Put

$$\begin{aligned}a_2 &= a_1 + x,\\a_3 &= a_2 + y,\\a_4 &= a_3 + z.\end{aligned}$$

Note that

$$\begin{aligned}a_2^2 &= a_1^2 + x^2 + 2a_1x,\\a_3^2 &= (a_1+x+y)^2\\&= a_1^2 + x^2 + y^2 + 2a_1x + 2a_1y + 2xy,\\a_4^2 &= (a_1+x+y+z)^2\\&= a_1^2 + x^2 + y^2 + z^2 + 2a_1x + 2a_1y + 2a_1z + 2xy + 2yz + 2zx\end{aligned}$$

hold. This gives

$$\begin{aligned}1 &= a_1^2 + a_2^2 + a_3^2 + a_4^2\\&= a_1^2\\&\quad + a_1^2 + x^2 + 2a_1x\\&\quad + a_1^2 + x^2 + y^2 + 2a_1x + 2a_1y + 2xy\end{aligned}$$

$$
\begin{aligned}
&+ a_1^2 + x^2 + y^2 + z^2 + 2a_1x + 2a_1y + 2a_1z + 2xy + 2yz + 2zx \\
&= 4a_1^2 + 3x^2 + 2y^2 + z^2 + 2a_1(3x + 2y + z) + 4xy + 2yz + 2zx \\
&= \left(2a_1 - \frac{3x + 2y + z}{2}\right)^2 - \frac{(3x + 2y + z)^2}{4} \\
&+ 3x^2 + 2y^2 + z^2 + 4xy + 2yz + 2zx \\
&= \left(2a_1 - \frac{3x + 2y + z}{2}\right)^2 - \left(\frac{9}{4}x^2 + y^2 + \frac{1}{4}z^2 + 3xy + yz + \frac{3}{2}zx\right) \\
&+ 3x^2 + 2y^2 + z^2 + 4xy + 2yz + 2zx \\
&= \left(2a_1 - \frac{3x + 2y + z}{2}\right)^2 + \frac{3}{4}x^2 + y^2 + \frac{3}{4}z^2 + xy + yz + \frac{1}{2}zx.
\end{aligned}
$$

If $x \geq \frac{1}{\sqrt{5}}, y \geq \frac{1}{\sqrt{5}}, z \geq \frac{1}{\sqrt{5}}$ holds, then we would obtain

$$
\begin{aligned}
1 &\geq \left(2a_1 - \frac{3x + 2y + z}{2}\right)^2 + \left(\frac{3}{4} + 1 + \frac{3}{4} + 1 + 1 + \frac{1}{2}\right)\frac{1}{5} \\
&= \left(2a_1 - \frac{3x + 2y + z}{2}\right)^2 + 1,
\end{aligned}
$$

which would imply

$$
a_1 = \frac{3x + 2y + z}{4} \geq \frac{3}{2\sqrt{5}},
$$

and hence,

$$
a_4 \geq \frac{3}{2\sqrt{5}} + \frac{3}{\sqrt{5}} = \frac{9}{2\sqrt{5}} > 1,
$$

which is impossible. This shows that at least one of x, y, z is less than $\frac{1}{\sqrt{5}}$. ■

§9 Algebraic substitutions

Example 9.1 (India RMO 2016a P2, India RMO 2016b P2)**.** Let a, b, c be positive real numbers such that

$$
\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.
$$

Prove that $abc \leq \frac{1}{8}$.

We refer to Example 5.4.

Solution 53. Put

$$
x = \frac{a}{1+a}, y = \frac{b}{1+b}, z = \frac{c}{1+c}.
$$

It follows that x, y, z are positive real numbers satisfying $x+y+z=1$. Note that

$$a = \frac{x}{1-x}, b = \frac{y}{1-y}, c = \frac{z}{1-z}.$$

We need to shows that $8abc \leq 1$, which is equivalent to $8xyz \leq (1-x)(1-y)(1-z)$. Using $x+y+z=1$, it follows that the above inequality is equivalent to

$$8xyz \leq (x+y)(y+z)(z+x),$$

which follows from the AM-GM inequality. ■

Remark. A careful reading of the above argument shows that the substitution

$$\frac{x}{x+y+z} = \frac{a}{1+a}, \frac{y}{x+y+z} = \frac{b}{1+b}, \frac{z}{x+y+z} = \frac{c}{1+c},$$

or equivalently, the substitution

$$a = \frac{x}{y+z}, b = \frac{y}{z+x}, c = \frac{z}{x+y}$$

reduces $8abc \leq 1$ to the inequality $8xyz \leq (x+y)(y+z)(z+x)$.

§10 Ravi substitution

In Fig. 1, the incircle touches the sides of the triangle ABC at the points D, E, F. Note that $AF = AE, BF = BD, CD = CE$ holds, and let us denote them by x, y, z respectively. This gives

$$a = BC = y + z, b = CA = z + x, c = AB = x + y$$

with $x, y, z > 0$.

Example 10.1 (India RMO 1996 P5)**.** Let ABC be a triangle and h_a the altitude through A. Prove that

$$(b+c)^2 \geq a^2 + 4h_a^2.$$

Solution 54. Since $\Delta = \frac{1}{2} a h_a$, the inequality reduces to

$$(b+c)^2 - a^2 \geq \frac{1}{a^2} 16\Delta^2,$$

which is equivalent to

$$a^2 \geq (c+a-b)(a+b-c).$$

This follows from the AM-GM inequality. ■

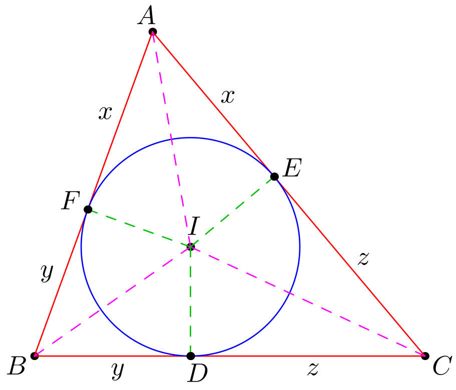

Figure 1: Ravi substitution

Remark. To prove the inequality $a^2 \geq (c+a-b)(a+b-c)$, one could substitute

$$a = y + z, b = z + x, c = x + y,$$

and note that the above inequality is equivalent to

$$(y+z)^2 \geq 4yz,$$

which holds since $(y-z)^2 \geq 0$.

Example 10.2 (India RMO 1999 P5)**.** If a, b, c are the sides of a triangle, prove the following inequality:

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3.$$

Solution 55. Let us use Ravi substitution, that is, let x, y, z be real numbers satisfying

$$a = y + z, b = z + x, c = x + y.$$

Note that x, y, z are positive by the triangle inequality. Observe that

$$\begin{aligned}
&\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \\
&= \frac{y+z}{2y} + \frac{z+x}{2z} + \frac{x+y}{2x} \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) \\
&\geq \frac{3}{2} + \frac{3}{2} \quad \text{(by the AM-GM inequality)} \\
&= 3.
\end{aligned}$$

■

Example 10.3 (India RMO 2001 P6)**.** If x, y, z are the sides of a triangle, then prove that

$$|x^2(y-z)+y^2(z-x)+z^2(x-y)| < xyz.$$

Solution 56. Note that

$$\begin{aligned} &x^2(y-z)+y^2(z-x)+z^2(x-y) \\ &= x^2y - zx^2 + y^2z - xy^2 + z^2x - yz^2 \\ &= (x^2-z^2)y + y^2(z-x) + zx(z-x) \\ &= (z-x)(y^2+zx-y(z+x)) \\ &= (z-x)(y-z)(y-x). \end{aligned}$$

Since

$$|x^2(y-z)+y^2(z-x)+z^2(x-y)|, xyz$$

are symmetric in x, y, z, without loss of generality, we may (and do) assume that $x \leq y \leq z$. Note that $|(z-x)(y-z)(y-x)| < xyz$ is immediate if any two of x, y, z are equal. It remains to consider the case that x, y, z are distinct, which we assume from now on. Let us use Ravi substitution, that is, let a, b, c be real numbers satisfying

$$x = b+c, y = c+a, z = a+b,$$

where a, b, c are positive by the triangle inequality. Using $x < y < z$, we obtain $a > b > c$. Note that

$$\begin{aligned} |(z-x)(y-z)(y-x)| &= (a-b)(b-c)(a-c) \\ &< aba \\ &< (a+c)(b+c)(a+b) \\ &= xyz, \end{aligned}$$

where the first inequality follows since $0 < a-b < a, 0 < b-c < b, 0 < a-c < c$ holds. ■

§11 Triangle inequality

Example 11.1. For real numbers x, y, z, show that

$$|x|+|y|+|z| \leq |x+y-z|+|y+z-x|+|z+x-y|.$$

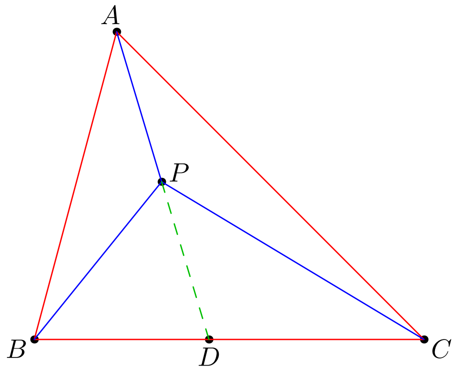

Figure 2: India RMO 1990 P5, Example 11.2

Solution 57. Put

$$x = \frac{b+c}{2}, y = \frac{c+a}{2}, z = \frac{a+b}{2}.$$

In other words, let a, b, c be real numbers defined by

$$a = y + z - x, b = z + x - y, c = x + y - z.$$

Applying the triangle inequality, we obtain

$$|a| + |b| \geq 2|z|, |b| + |c| \geq 2|x|, |c| + |a| \geq 2|y|.$$

The result follows by adding them. ■

Example 11.2 (India RMO 1990 P5)**.** Let P be a point inside a triangle ABC. Let s denote the semiperimeter $\frac{1}{2}(AB + BC + CA)$. Prove that

$$s < AP + BP + CP < 2s.$$

Solution 58. Applying the triangle inequality to PBC, we get $BC < BP+CP$. Similarly, the inequalities $CA < CP + AP, AB < AP + BP$ follow. Adding them yields

$$AB + BC + CA < 2(AP + BP + CP),$$

or equivalently, $s < AP + BP + CP$ holds. Let D denote the point on BC such that A, P, D are collinear. Note that

$$\begin{aligned} AP + BP &< AP + BD + DP \quad \text{(applying the triangle inequality to } BDP) \\ &= AD + BD \\ &< AC + DC + BD \quad \text{(applying the triangle inequality to } ADC) \end{aligned}$$

$$= AC + BC$$

hold. Similarly, it follows that

$$BP + CP < AB + CA, CP + AP < AB + BC.$$

Adding these inequalities, we obtain

$$AP + BP + CP < AB + BC + CA = 2s.$$

■

Example 11.3 (India RMO 1995 P5)**.** Show that for any triangle ABC, the following inequality is true:

$$a^2 + b^2 + c^2 > \sqrt{3}\max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\}.$$

Solution 59. Since the above inequality is symmetric with respect to a, b, c, without loss of generality, we assume that $a > b > c$. Thus we need to show that

$$a^2 + b^2 + c^2 > \sqrt{3}(a^2 - c^2)$$

holds. Using the triangle inequality, we obtain $b > a - c > 0$. This yields

$$\begin{aligned}
a^2 + b^2 + c^2 - \sqrt{3}(a^2 - c^2) &> a^2 + (a - c)^2 + c^2 - \sqrt{3}(a^2 - c^2) \\
&= 2(a^2 + c^2 - ca) - \sqrt{3}(a^2 - c^2) \\
&= (2 - \sqrt{3})a^2 + (2 + \sqrt{3})c^2 - 2ca \\
&= (2 - \sqrt{3})(a^2 - 2ac(2 + \sqrt{3}) + (2 + \sqrt{3})^2) \\
&= (2 - \sqrt{3})(a - (2 + \sqrt{3})c)^2 \\
&\geq 0.
\end{aligned}$$

This completes the proof. ■

Example 11.4 (India RMO 1997 P5)**.** Let x, y and z be three distinct real positive numbers. Determine with proof whether or not the three real numbers

$$\left|\frac{x}{y} - \frac{y}{x}\right|, \left|\frac{y}{z} - \frac{z}{y}\right|, \left|\frac{z}{x} - \frac{x}{z}\right|$$

can be the lengths of the sides of a triangle.

Solution 60. Without loss of generality, let us assume that $x > y > z$. Note that

$$z(x^2 - y^2) + x(y^2 - z^2) - y(x^2 - z^2)$$

$$
\begin{aligned}
&= zx(x-z) + (x-z)y^2 - y(x^2 - z^2) \\
&= (x-z)(zx + y^2 - y(x+z)) \\
&= (x-z)(y-x)(y-z) \\
&< 0.
\end{aligned}
$$

By the triangle inequality, it follows that the given three numbers cannot be the lengths of the sides of a triangle. ■

Example 11.5 (India RMO 2009 P5)**.** A convex polygon is such that the distance between any two vertices does not exceed 1.

(i) Prove that the distance between any two points on the boundary of the polygon does not exceed 1.

(ii) If X and Y are two distinct points inside the polygon, prove that there exists a point Z on the boundary of the polygon such that $XZ + YZ \leq 1$.

Solution 61. Note that for any four complex numbers z_1, z_2, z_3, z_4 and any $0 \leq t, s \leq 1$, we have

$$
\begin{aligned}
&(tz_1 + (1-t)z_2) - (sz_3 + (1-s)z_4) \\
&= t(z_1 - z_3) + (t-s)z_3 + (1-s)(z_2 - z_4) + (s-t)z_2 \\
&= t(z_1 - z_3) + (s-t)(z_3 - z_2) + (1-s)(z_2 - z_4)
\end{aligned}
$$

and

$$
\begin{aligned}
&(tz_1 + (1-t)z_2) - (sz_3 + (1-s)z_4) \\
&= s(z_1 - z_3) + (t-s)z_1 + (1-t)(z_2 - z_4) + (s-t)z_4 \\
&= s(z_1 - z_3) + (t-s)(z_1 - z_4) + (1-t)(z_2 - z_4).
\end{aligned}
$$

If the distance between no two of z_1, z_2, z_3, z_4 exceeds 1, then

$$
|(tz_1 + (1-t)z_2) - (sz_3 + (1-s)z_4)| \leq \begin{cases} t + (s-t) + (1-s) = 1 & \text{if } s \geq t, \\ s + (t-s) + (1-t) = 1 & \text{if } t \geq s. \end{cases}
$$

This proves part (i).

Extending XY to the boundary of the polygon, we find two points Z_1, Z_2 on the boundary such that Z_1, X, Y, Z_2 are collinear. Note that

$$
\begin{aligned}
(XZ_1 + YZ_1) + (XZ_2 + YZ_2) &= (XZ_1 + XZ_2) + (YZ_1 + YZ_2) \\
&= 2Z_1Z_2 \\
&\leq 2,
\end{aligned}
$$

which shows that $XZ_1 + YZ_1 \leq 1$ or $XZ_2 + YZ_2 \leq 1$ holds. This proves part (ii). ■

Example 11.6 (India RMO 2014e P1)**.** Three positive real numbers a, b, c are such that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. Can a, b, c be the lengths of the sides of a triangle? Justify your answer.

Solution 62. The given condition is equivalent to

$$(a - 2b)^2 + (b - 2c)^2 = 0.$$

Since a, b, c are real numbers, this gives $b = 2c, a = 2b = 2c$. Note that $b + c = 3c < 2c = a$ holds. So by the triangle inequality, a, b, c cannot be the lengths of the sides of a triangle. ■

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[**PK74**] G. Pólya and J. Kilpatrick. *The Stanford Mathematics Problem Book: With Hints and Solutions.* Dover books on mathematics. Teachers College Press, 1974. isbn: 9780486469249. url: https://books.google.de/books?id=Q8Gn51gS6RoC (cited p. 18)