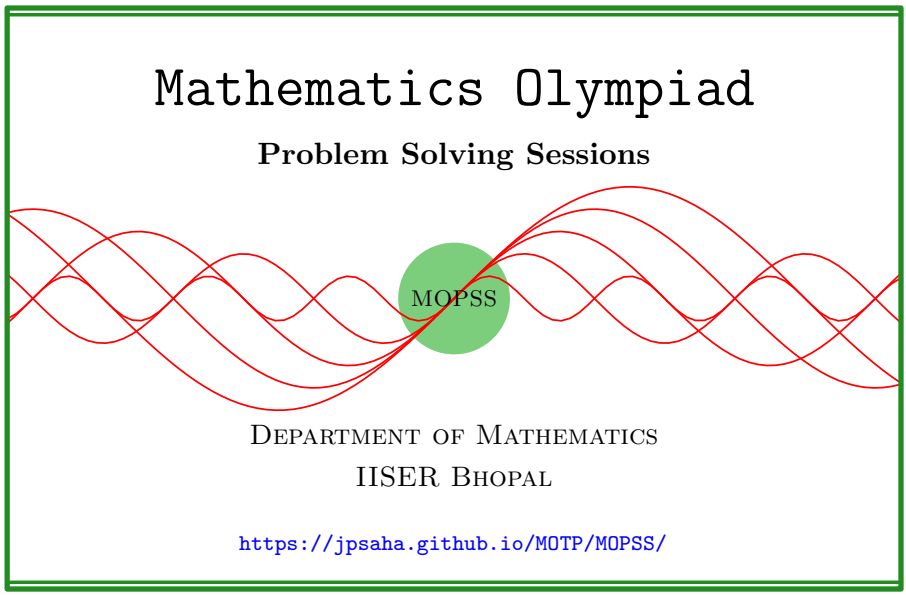
Induction

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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§1 Induction

Exercise 1 (All-Russian Mathematical Olympiad 1989, , N. Agahanov)**.** [**Sli97**, p. 14]

1. Do there exist real numbers a and b such that $a+b$ is rational and a^n+b^n is irrational for each natural number $n \geq 2$?
2. Do there exist real numbers a and b such that $a+b$ is irrational and a^n+b^n is rational for each natural number $n \geq 2$?

Walkthrough —

(a)

Solution 1. ■

Exercise 2 (Japan Mathematical Olympiad 1993 P2)**.** For a positive integer n, let $d(n)$ denote its largest odd divisor. Define

$$D(n) = d(1) + d(2) + \cdots + d(n),$$
$$T(n) = 1 + 2 + \cdots + n.$$

Show that there exist infinitely many positive integers n such that $3D(n) = 2T(n)$.

Walkthrough —

(a) Check that $n = 2, 6, 14, 30$ works.

(b) This may suggest that $n = 2^m - 2$ works for all $m \geq 2$.

(c) Does induction help?

Solution 2. Let us establish the following claim.

Claim — For any positive integer n,

$$D(2^n) = 2^n + \frac{1}{3}(2^n - 1)(2^n - 2)$$

holds.

Proof of the Claim. We prove the claim by induction on n. Note that

$$\begin{aligned} D(2) &= d(1) + d(2) \\ &= 1 + 1, \\ &= 2 + \frac{1}{3}(2-1)(2-2), \\ D(2^2) &= D(2) + d(3) + d(4) \\ &= 2 + 3 + 1 \\ &= 6 \\ &= 2^2 + \frac{1}{3}(2^2 - 1)(2^2 - 2) \end{aligned}$$

hold. Now, let $k \geq 2$ be a positive integer, and assume that the claim holds for for all positive integers up to k. Note that

$$\begin{aligned} & D(2^{k+1}) \\ &= D(2^k) + \sum_{i=1}^{2^k} d(2^k + i) \\ &= D(2^k) + \sum_{i=1}^{2^{k-1}} d(2^k + 2i - 1) + \sum_{i=1}^{2^{k-1}} d(2^k + 2i) \\ &= D(2^k) + \sum_{i=1}^{2^{k-1}} (2^k + 2i - 1) + \sum_{i=1}^{2^{k-1}} d(2^{k-1} + i) \\ &= D(2^k) + \frac{2^{k-1}}{2} \cdot (2^k + 1 + 2^k + 2 \cdot 2^{k-1} - 1) + D(2^k) - D(2^{k-1}) \\ &= 2D(2^k) - D(2^{k-1}) + 2^{k-2}(2^k + 2^{k+1}) \\ &= 2\left(2^k + \frac{1}{3}(2^k - 1)(2^k - 2)\right) \\ &\quad - \left(2^{k-1} + \frac{1}{3}(2^{k-1} - 1)(2^{k-1} - 2)\right) + 2^{k-2}(2^k + 2^{k+1}) \\ &= 2^{k+1} + \frac{1}{3}\left(2(2^k - 1)(2^k - 2) - (2^{k-1} - 1)(2^{k-1} - 2)\right) \\ &\quad + 2^{k-2}(2^k + 2^{k+1}) - 2^{k-1} \end{aligned}$$

$$= 2^{k+1} + \frac{1}{3}\left(2(2x-1)(2x-2) - (x-1)(x-2) + \frac{3}{2}x(2x+4x)\right) - x$$

(where $x = 2^{k-1}$)

$$= 2^{k+1} + \frac{1}{3}\left(8x^2 - 12x + 4 - x^2 + 3x - 2 + 9x^2\right) - x$$

$$= 2^{k+1} + \frac{1}{3}(16x^2 - 12x + 2)$$

$$= 2^{k+1} + \frac{1}{3}(4x-1)(4x-2)$$

$$= 2^{k+1} + \frac{1}{3}(2^{k+1} - 1)(2^{k+1} - 2).$$

□

For any integer $m \geq 2$, using the claim, we have

$$\begin{aligned} D(2^m - 2) &= D(2^m) - d(2^m - 1) - d(2^m) \\ &= D(2^m) - (2^m - 1) - 1 \\ &= 2^m + \frac{1}{3}(2^m - 1)(2^m - 2) - 2^m \\ &= \frac{1}{3}(2^m - 1)(2^m - 2), \end{aligned}$$

and consequently, we obtain

$$\begin{aligned} 3D(2^m - 2) &= (2^m - 1)(2^m - 2) \\ &= 2(1 + 2 + \cdots + (2^m - 2)) \\ &= 2T(2^m - 2). \end{aligned}$$

■

Exercise 3 (All-Russian Mathematical Olympiad 2007 Grade 10 Day 1 P2, , by A. Khrabrov)**.** Consider a polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Let m denote the minimum of the integers

$$a_0, a_0 + a_1, \ldots, a_0 + a_1 + \cdots + a_n.$$

Prove that $P(x) \geq mx^n$ for all $x \geq 1$.

Walkthrough —

(a)

Solution 3. Note that

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

$$= a_0(x^n - x^{n-1}) + (a_0 + a_1)(x^{n-1} - x^{n-2}) \\ + \cdots + (a_0 + a_1 + \cdots + a_{n-1})(x - 1) + (a_0 + a_1 + \cdots + a_n)$$

holds, which implies that

$$\begin{aligned} &P(x) - mx^n \\ &= a_0(x^n - x^{n-1}) + (a_0 + a_1)(x^{n-1} - x^{n-2}) \\ &\quad + \cdots + (a_0 + a_1 + \cdots + a_{n-1})(x - 1) + (a_0 + a_1 + \cdots + a_n) \\ &\quad - m(x^n - x^{n-1}) - m(x^{n-1} - x^{n-2}) - \cdots - m(x - 1) - m \\ &= (a_0 - m)(x^n - x^{n-1}) + (a_0 + a_1 - m)(x^{n-1} - x^{n-2}) \\ &\quad + \cdots + (a_0 + a_1 + \cdots + a_{n-1} - m)(x - 1) + (a_0 + a_1 + \cdots + a_n - m). \end{aligned}$$

It follows that for any $x \geq 1$,

$$P(x) \geq mx^n.$$

■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Sli97**] A.M. Slinko. *USSR Mathematical Olympiads, 1989-1992*. Australian Mathematics Trust enrichment series. Australian Mathematics Trust, 1997. ISBN: 9780646336183. URL: https://books.google.de/books?id=OVlnomt12GQC (cited p. 2)