Crossing the x-axis

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Crossing the x-axis

Here are a few problems from this notes, and this one.

Example 1.1. Suppose $P(x)$ is a polynomial with real coefficients such that $P(x) = x$ has no real solution. Show that $P(P(x)) = x$ has no real solutions.

Solution 1. Since $x \mapsto P(x)$ defines a continuous map from $\mathbb{R} \to \mathbb{R}$, by the intermediate value theorem, it follows that $P(x) > x$ holds for all $x \in \mathbb{R}$ or $P(x) < x$ holds for all $x \in \mathbb{R}$. If $P(x) > x$ holds for all $x \in \mathbb{R}$, then $P(P(x)) > P(x) > x$ holds for all $x \in \mathbb{R}$, and hence $P(P(x)) = x$ has no real solutions. Similarly, if $P(x) < x$ holds for all $x \in \mathbb{R}$, then $P(P(x)) = x$ has no real solutions. ■

Example 1.2. Any polynomial of odd degree with real coefficients has a real root.

Example 1.3. Let $P(x)$ and $Q(x)$ be monic polynomials of degree 10 having real coefficients. Assume that the equation $P(x) = Q(x)$ has no real roots. Prove that the equation $P(x + 1) = Q(x - 1)$ has at least one real root.

Solution 2. Note that $P(x) - Q(x)$ is a polynomial of degree at most 9 having real coefficients. Since $P(x) - Q(x)$ has no real root, it follows that it has degree at most 8. In other words, the coefficients of x^9 in $P(x), Q(x)$ are the same. Note that $P(x+1) - Q(x-1)$ is of degree ≤ 9, and the coefficient of x^9 in $P(x+1) - Q(x-1)$ is equal to the coefficient of x^9 in $(x+1)^{10} - (x-1)^{10}$, which is equal to 20. This shows that $P(x+1) - Q(x-1)$ is a polynomial of degree 9 with real coefficients. Consequently, it has at least one real root. ■

Example 1.4. Let $P(x)$ be a nonconstant polynomial with real coefficients having a real root. Suppose it does not vanish at 0. Show that the monomial terms appearing in $P(x)$ can be erased one by one to obtain its constant term such that the intermediate polynomial have at least one real root.

Solution 3. Write $P(x) = a_n x^n + \cdots + a_0$ with $a_n \ldots, a_0$ lying in $\mathbb{R}$, and $a_n a_0 \neq 0$. Since any polynomial of odd degree has a real root, it follows that if the degree of $P(x)$ is odd, then the nonconstant monomials, other than the leading term, can be erased one by one, and then, the leading term can be erased to obtain a_0, and the intermediate polynomials have a real root. If a_n, a_0 are of opposite signs and n is even, then the same process can be followed, and note that any of the intermediate polynomials takes values of opposite signs at 0 and at a large enough integer, and hence has a real root.

Let us assume that n is odd, and a_n, a_0 are of the same sign. Let α denote a real root of $P(x)$. Let $Q(x)$ denote the polynomial $P(x) - a_n x^n$. Note that $Q(0) = a_0$ and $Q(\alpha) = -a_n\alpha^n$. Since a_0 is nonzero, it follows that $\alpha \neq 0$, and hence, $Q(0), Q(\alpha)$ are of opposite signs. This shows that $Q(x)$ is a nonconstant polynomial with real coefficients having a real root, and it does not vanish at 0. Since the degree of $Q(x)$ is smaller than that of $P(x)$, by induction, we are done. ■

Example 1.5 (China TST 1995). Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \cdots + \square x + 1.$$

They take turns to fill the empty boxes. If the resulting polynomial has no real root, Alice wins, otherwise, Bob wins. If Alice goes first, who has a winning strategy?

Solution 4. Bob has a winning strategy, as described below.

Bob makes sure that at the end of each of his turns except the last one, the number of even powers of x whose coefficients have been provided by some of them is equal to the number of odd powers of x whose coefficients have been provided by some of them. This can be done, for instance, if during a turn of Bob, other than the last turn, Bob provides the coefficient of an odd (resp. even) power of x if Alice has provided the coefficient of an even (resp. odd) power of x in the preceeding turn.

Since $n \geq 2$, it follows that Bob gets at least one turn. At the beginning of the final turn of Bob, there are two powers of x whose coefficients are to be determined, denote them by x^i, x^j, their coefficients by c_i, c_j respectively. Let $Q(x)$ denote the polynomial, obtained by the erasing the terms corresponding to x^i, x^j from the polynomial that Bob had at the beginning of his final turn. Note that

$$P(x) = Q(x) + c_i x^i + c_j x^j.$$

Note that at least of i, j is odd. Interchanging i, j if necessary, let us assume that i is odd. We descrie the strategy that Bob follows in the two cases below.

Let us consider the case that j is even. Bob determines c_j in such a way that for any choice of c_i, the completed polynomial $P(x)$ is guaranteed to have at least one real root. This can be done, for instance, by taking c_j satisfying

$$Q(1) + Q(-1) + 2c_j = 0.$$

For any choice of c_i, the above choice of c_j shows that $P(1)+P(-1)=0$, which implies that $P(x)$ has a root in the interval $[-1,1]$.

Let us consider the case that j is odd. Bob determines c_j in such a way that for any choice of c_i by Alice in the next turn, the completed polynomial $P(x)$ is guaranteed to have at least one real root. This can be done, for instance, by taking c_j satisfying

$$Q(2)+c_j2^j+2^iQ(-1)-c_j2^i=0.$$

Since $i \neq j$, the above holds for some $c_j \in \mathbb{R}$. For any choice of c_i, note that

$$P(2)+2^iP(-1)=0$$

holds, which implies that $P(x)$ has a root in $[-1,2]$. ■

 The content posted here and at this blog by Evan Chen are quite useful.