Growth of polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 On the growth of polynomials

Example 1.1 (India BStat-BMath 2012)**.** Show that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real root.

> **Summary** — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

Solution 1. Let α be a real number. Let us consider the following cases.

1. $\alpha \geq 1$,
2. $\alpha \leq 0$,
3. $0 \leq \alpha \leq 1$.

If $\alpha \geq 1$, then

$$\begin{aligned} &\alpha^8 - \alpha^7 + \alpha^2 - \alpha + 15 \\ &= \alpha^7(\alpha - 1) + \alpha(\alpha - 1) + 15 \\ &\geq 15. \end{aligned}$$

If $\alpha \leq 0$, then

$$\begin{aligned} &\alpha^8 - \alpha^7 + \alpha^2 - \alpha + 15 \\ &= \alpha^8 + (-\alpha^7) + \alpha^2 + (-\alpha) + 15 \\ &\geq 15. \end{aligned}$$

If $0 \leq \alpha \leq 1$, then

$$\begin{aligned} &\alpha^8 - \alpha^7 + \alpha^2 - \alpha + 15 \\ &= \alpha^8 + (1 - \alpha^7) + \alpha^2 + (1 - \alpha) + 13 \\ &\geq 13. \end{aligned}$$

It follows that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real root. ■

Example 1.2. Does there exist a polynomial $P(x)$ with rational coefficients such that $\sin x = P(x)$ for all $x \geq 100$?

Solution 2. Suppose there exists a polynomial $P(x)$ with rational coefficients such that $\sin x = P(x)$ for all $x \geq 100$. It follows that $P(x)$ has absolute value at most 1 for all $x \geq 100$.

> **Claim —** Let $f(x)$ be a nonconstant polynomial with real coefficients. Then for any given $M > 0$, there exists a real number $x_0 > 0$ such that
>
> $$|f(x)| > M$$
>
> for all real number x satisfying $|x| > x_0$.

Proof of the Claim. Write

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0,$$

where $a_d, a_{d-1}, \ldots, a_0$ lie in $\mathbb{R}$ and d denotes the degree of $f(x)$. Note that for any real α,

$$\begin{aligned}
|f(\alpha)| &= \left|a_d\alpha^d + a_{d-1}\alpha^{d-1} + \cdots + a_1\alpha + a_0\right| \\
&\geq \left|a_d\alpha^d\right| - \left|a_{d-1}\alpha^{d-1} + \cdots + a_1\alpha + a_0\right| \\
&\geq \left|a_d\alpha^d\right| - \left|a_{d-1}\alpha^{d-1}\right| - \cdots - |a_1\alpha| - |a_0| \\
&= \left(\frac{1}{d}\left|a_d\alpha^d\right| - \left|a_{d-1}\alpha^{d-1}\right|\right) \\
&+ \left(\frac{1}{d}\left|a_d\alpha^d\right| - \left|a_{d-2}\alpha^{d-2}\right|\right) \\
&+ \cdots + \left(\frac{1}{d}\left|a_d\alpha^d\right| - |a_1\alpha|\right) \\
&+ \left(\frac{1}{d}\left|a_d\alpha^d\right| - |a_0|\right) \\
&= \frac{|a_d|}{d}\left|\alpha^{d-1}\right|\left(|\alpha| - \left|\frac{da_{d-1}}{a_d}\right|\right) \\
&+ \frac{|a_d|}{d}\left|\alpha^{d-2}\right|\left(|\alpha|^2 - \left|\frac{da_{d-2}}{a_d}\right|\right) \\
&+ \cdots + \frac{|a_d|}{d}|\alpha|\left(|\alpha|^{d-1} - \left|\frac{da_1}{a_d}\right|\right) \\
&+ \frac{|a_d|}{d}\left(|\alpha|^d - \left|\frac{da_0}{a_d}\right|\right).
\end{aligned}$$

Hence, for any given $M > 0$,

$$|f(x)| > M$$

holds for any real number x of large enough absolute value. Indeed, for any real number x satisfying

$$
\begin{aligned}
|x| &> \left|\frac{da_{d-1}}{a_d}\right|, \\
|x|^2 &> \left|\frac{da_{d-2}}{a_d}\right|, \\
\cdots &> \cdots, \\
|x|^{d-1} &> \left|\frac{da_1}{a_d}\right|, \\
|x|^d &> \left|\frac{da_0}{a_d}\right| + \frac{Md}{|a_d|},
\end{aligned}
$$

or equivalently, satisfying

$$x > \max\left\{\left|\frac{da_{d-1}}{a_d}\right|, \left(\left|\frac{da_{d-2}}{a_d}\right|\right)^{1/2}, \dots, \left(\left|\frac{da_1}{a_d}\right|\right)^{1/(d-1)}, \left(\left|\frac{da_0}{a_d}\right| + \frac{Md}{|a_d|}\right)^{1/d}\right\},$$

the inequality $|f(x)| > M$ holds. The Claim follows by taking

$$x_0 = \max\left\{\left|\frac{da_{d-1}}{a_d}\right|, \left(\left|\frac{da_{d-2}}{a_d}\right|\right)^{1/2}, \dots, \left(\left|\frac{da_1}{a_d}\right|\right)^{1/(d-1)}, \left(\left|\frac{da_0}{a_d}\right| + \frac{Md}{|a_d|}\right)^{1/d}\right\}.$$

□

By the above Claim, it follows that $P(x)$ is a constant polynomial. This shows that $\sin x$ is constant on the interval $[100, \infty)$, which is impossible since $\sin 100\pi \neq \sin 101\pi$ and $101\pi, 100\pi$ lie in $[100, \infty)$. ■

Example 1.3 (India RMO 2015b P3). Let $P(x)$ be a polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

> **Summary** — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

Solution 3. Note that $P(x) = 1$ serves as a counterexample. Henceforth, let us assume that $P(x)$ is a nonconstant polynomial.

Let $Q(x), R(x)$ be polynomials with rational coefficients such that

$$P(P(x) - 2015) = P(x)Q(x) + R(x)$$

and $R(x) = 0$ or $\deg R(x) \leq \deg P(x)$. Note that $P(n)$ is positive for all integer $n \geq 1$ since the coefficients of $P(x)$ are positive integers. By the given condition, it follows that $P(n)$ divides $R(n)$ for any integer $n \geq 1$.

Claim — Let $f(x), g(x)$ be two polynomials with real coefficients. Suppose $f(x)$ is a nonconstant polynomial with a positive leading coefficient, and $\deg g(x) < \deg f(x)$. Then there exists an integer $n_0 \geq 1$ such that

$$f(n) > g(n)$$

for any $n \geq n_0$.

Proof of the Claim. Note that it suffices to proves the Claim if $f(x)$ is a monomial, that is, a power of x. Indeed, write $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ with $a_0, \ldots, a_d \in \mathbb{R}$ and d denoting the degree of f. Also write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. Noting that $a_d > 0$, it follows that for a positive integer n, the inequality

$$a_d n^d + a_{d-1}n^{d-1} + \cdots + a_0 > b_e n^e + b_{e-1}n^{e-1} + \cdots + b_0$$

holds if

$$a_d n^d > \frac{a_{d-1}}{a_d} n^{d-1} + \cdots + \frac{a_0}{a_d} + \frac{b_e}{a_d} n^e + \frac{b_{e-1}}{a_d} n^{e-1} + \cdots + \frac{b_0}{a_d}$$

is satisfied, which can be concluded provided the Claim is known in the case when f is a monomial.

Let us assume that f is a monomial. Write $f(x) = x^d$ where d denotes the degree of f, and write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. For any integer n, note that

$$\begin{aligned} f(n) - g(n) &= \left(\frac{1}{e+1}n^d - b_e n^e\right) \\ &\quad + \left(\frac{1}{e+1}n^d - b_{e-1} n^{e-1}\right) \\ &\quad + \cdots + \left(\frac{1}{e+1}n^d - b_0\right) \\ &\geq \left(\frac{1}{e+1}n^d - |b_e| n^e\right) \\ &\quad + \left(\frac{1}{e+1}n^d - |b_{e-1}| n^{e-1}\right) \\ &\quad + \cdots + \left(\frac{1}{e+1}n^d - |b_0|\right). \end{aligned}$$

Since $d \geq e$, it follows that there exists an integer $n_0 \geq 1$ such that

$$\frac{1}{e+1}n^d - |b_e| n^e \frac{1}{e+1}n^d - |b_{e-1}| n^{e-1}, \ldots, \frac{1}{e+1}n^d - |b_0|$$

are positive for any $n \geq n_0$. This proves the Claim. ☐

By the above Claim, it follows that $R(x)$ is the zero polynomial. This implies that

$$P(P(x) - 2015) = P(x)Q(x).$$

Since $P(x)$ is a nonconstant polynomial, it has a root z in $\mathbb{C}$. Substituting $x = z$ yields

$$P(-2015) = 0.$$

■