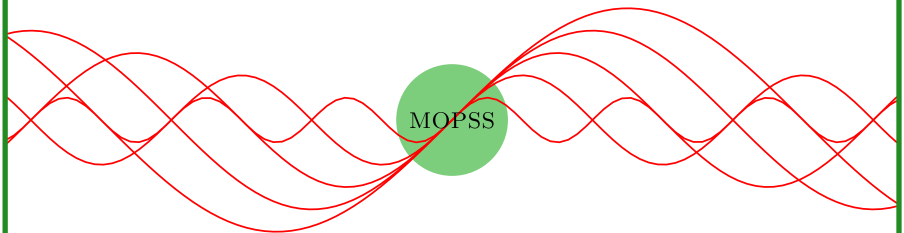
Functions

MOPSS

Mathematics Olympiad

Problem Solving Sessions



Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Functions

Exercise 1.1 (All-Russian Mathematical Olympiad 2014 Grade 10 Day 1 P2, AoPS, by O. Podlipsky)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x)^2 \leq f(y)$$

holds for any $x, y \in \mathbb{R}$ with $x > y$. Show that

$$0 \leq f(x) \leq 1$$

holds for any $x \in \mathbb{R}$.

Walkthrough —

(a) Show that $f(x) \geq 0$ for any $x \in \mathbb{R}$.

(b) Using the given inequality, show that for any $x, y \in \mathbb{R}$ with $x > y$, and for any positive integer n, the inequality

$$f(x)^{2^n} \leq f(y)$$

holds.

(c) Use the above to conclude that $f(x) \leq 1$ for any $x \in \mathbb{R}$.

Solution 1. Note that for any $x \in \mathbb{R}$, we have

$$f(x+1)^2 \leq f(x),$$

which shows that $f(x) \geq 0$.

To show that $f(x) \leq 1$ for any $x \in \mathbb{R}$, let us establish the following claim.

Claim — For any $x, y \in \mathbb{R}$ with $x > y$, and for any positive integer n, the inequality

$$f(x)^{2^n} \leq f(y)$$

holds.

Proof of the Claim. Note that the inequality holds for $n = 1$ by hypothesis. Let us assume that n is a positive integer such that the inequality holds for any $x, y \in \mathbb{R}$ with $x > y$. Then, for any $x, y \in \mathbb{R}$ with $x > y$, we have

$$x > \frac{x+y}{2} > y,$$

which implies that

$$f(x)^{2^n} \leq f\left(\frac{x+y}{2}\right) \quad \text{and} \quad f\left(\frac{x+y}{2}\right)^2 \leq f(y).$$

Combining these two inequalities and using that $f(x)$ is nonnegative, we obtain

$$f(x)^{2^{n+1}} \leq f(y).$$

By the principle of mathematical induction, the claim follows. □

Let x be an arbitrary real number. Using the above claim, we obtain

$$f(x)^{2^n} \leq f(x-1)$$

for **any** positive integer n. If $f(x) \leq 1$, then we are done. It remains to consider the case that $f(x) > 1$, which we assume from now on. It follows that

$$f(x-1) \geq 1 + 2^n\left(f(x) - 1\right)$$

for **any** positive integer n. This is impossible since $f(x) - 1$ is positive.

Combining the above, we conclude that

$$0 \leq f(x) \leq 1$$

holds for any $x \in \mathbb{R}$. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)