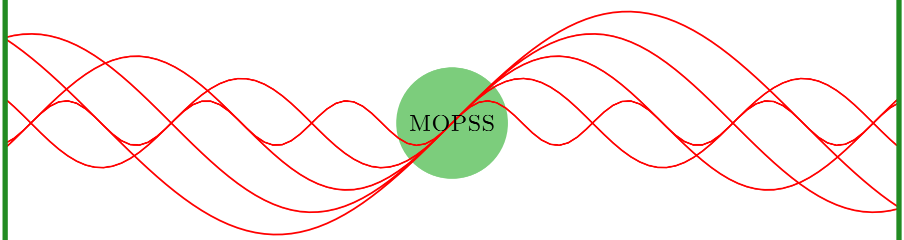
Finite differences

MOPSS

6 March 2025

Mathematics Olympiad

Problem Solving Sessions



Department of Mathematics

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Finite differences

Note that

$$k^2 - (k-1)^2 = 2k - 1$$

holds for any integer k. In particular, given a positive integer n, we have

$$\begin{aligned} 2^2 - 1^2 &= 2 \cdot 2 - 1, \\ 3^2 - 2^2 &= 2 \cdot 3 - 1, \\ 4^2 - 3^2 &= 2 \cdot 4 - 1, \\ \cdots &= \cdots, \\ n^2 - (n-1)^2 &= 2 \cdot n - 1. \end{aligned}$$

Adding them, it follows that

$$n^2 - 1 = 2(2 + 3 + \cdots + n) - (n-1),$$

which yields

$$\boxed{1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}}.$$

Consider the polynomial $P(x) = x^3$. Note that

$$P(k) - P(k-1) = 3k^2 - 3k + 1.$$

Using a similar argument as above, it follows that given a positive integer n, we have

$$\begin{aligned} 1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1, \\ 2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1, \\ 3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1, \\ 4^3 - 3^3 &= 3 \cdot 4^2 - 3 \cdot 4 + 1, \\ \cdots &= \cdots, \\ n^3 - (n-1)^3 &= 3 \cdot n^2 - 3 \cdot n + 1. \end{aligned}$$

Adding them, it follows that

$$n^3 = 3(1 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n,$$

which yields

$$\begin{aligned} 1+2^2+3^2+\cdots+n^2 &= \frac{n^3-n}{3}+(1+2+3+\cdots+n) \\ &= \frac{n^3-n}{3}+\frac{n(n+1)}{2} \\ &= \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Example 1.1 (India RMO 2013b P3)**.** Consider the expression

$$2013^2+2014^2+2015^2+\cdots+n^2.$$

Prove that there exists a natural number $n>2013$ for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

Summary — "Differentiating" a polynomial enough times makes it linear.

Walkthrough —

(a) Consider the polynomial $P(k)=k^2$, and the polynomial $Q(k):=P(k)-(k-1)$.

(b) Since $Q(k)$ is a linear polynomial in k, the difference $R(k):=Q(k)-Q(k-2)$ is a constant, that is, it does not depend on k.

(c) Note that $R(k)$ is a **± 1-linear combination**[a] of four consecutive squares.

(d) Does this help?

[a]What does it mean?

Solution 1. Consider the polynomial $P(k)=k^2$, and the polynomial $Q(k):=P(k)-(k-1)$. Since $Q(k)$ is a linear polynomial in k, the difference $R(k):=Q(k)-Q(k-2)$ is a constant, that is, it does not depend on k. Indeed, $Q(k)=2k-1$, and $R(k)=4$. Note that $R(k)=k^2-(k-1)^2-(k-2)^2+(k-3)^2$.

Note that

$$2013^2+2014^2+2015^2+2016^2+2017^2>9999$$

holds, and the integers $2013^2+2014^2+2015^2+2016^2+2017^2$, 9999 are congruent modulo 4, that is, they differ by a multiple of 4. Let $m\geq 1$ be an integer such that

$$9999=2013^2+2014^2+2015^2+2016^2+2017^2-4m$$

holds. Since

$$-k^2+(k+1)^2+(k+2)^2-(k+3)^2=-4,$$

it follows that

$$\begin{aligned}
&9999 \\
&= 2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2 \\
&- 2018^2 + 2019^2 + 2020^2 - 2021^2 \\
&- \dots \\
&- \big((2018 + 4(m-1))^2 + (2019 + 4(m-1))^2 \\
&\qquad + (2020 + 4(m-1))^2 - (2021 + 4(m-1))^2\big).
\end{aligned}$$

It follows that there exists a natural number $n = 2021 + 4(m-1) > 2013$, for which one can change a suitable number of plus signs to minus signs in the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2$$

to make the resulting expression equal to 9999. ■

Example 1.2 (Bay Area MO 12 2016 P4)**.** Find a positive integer N and $a_1, a_2, \dots, a_N$, where $a_k = 1$ or $a_k = -1$ for each $k = 1, 2, \dots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_N \cdot N^3 = 20162016,$$

or show that this is impossible.

Summary — "Differentiating" a polynomial enough times makes it linear.

Walkthrough —

(a) Consider the polynomial $P(k) := k^3$. Note that $R(k) := P(k) - P(k-1)$ is a quadratic polynomial in k.

(b) Also note that $S(k) := R(k) - R(k-2)$ is a linear polynomial in k.

Solution 2. Consider the polynomial $P(k) := k^3$. Note that $R(k) := P(k) - P(k-1)$ is equal to $3k^2 - 3k + 1$. Also note that $S(k) := R(k) - R(k-2)$ is equal to $6(2k-2) - 6$. This gives $S(k) - S(k-4) = 48$. It follows that **some ± 1-linear combination** of any given eight consecutive cubes is equal to 48. More specifically,

$$k^3 - (k-1)^3 - (k-2)^3 + (k-3)^3 - (k-4)^3 + (k-5)^5 + (k-6)^3 - (k-7)^3 = 48,$$

or equivalently,

$$-k^3 + (k+1)^3 + (k+2)^3 - (k+3)^3 - (k+4)^3 + (k+5)^3 + (k+6)^3 - (k+7)^3 = 48.$$

Note that 20162016 is divisible by 3 and 16. Since $3, 16$ do not have any common prime factor, it follows that 20162016 is a multiple of 48. Write

$$f(k) = -k^3 + (k+1)^3 + (k+2)^3 - (k+3)^3 - (k+4)^3 + (k+5)^3 + (k+6)^3 - (k+7)^3.$$

Note that

$$f(1) + f(9) + f(17) + \cdots + f(8m-7) = 20162016,$$

where m denotes the integer $20162016/48$. We conclude that one may take $N = 8m = 20162016/6 = 3360336$ so that the given condition holds. ■