Differentiation and multiple roots

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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§1 Differentiation and multiple roots

Lemma 1

Let $P(x)$ be a polynomial with complex coefficients, and α be a complex number. Then α is a root of $P(x)$ having multiplicity at least $r \geq 2$ (i.e., $(x-\alpha)^r$ divides $P(x)$) if and only if it is a root of $P(x), P'(x), \ldots, P^{(r)}(x)$, where $P^{(r)}(x)$ denotes the r-fold derivative of $P(x)$.

Example 1.1 (Putnam 1956 B7, IMOSL 1981 Cuba)**.** The polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeroes but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. **Assume that at least one of $P(z), Q(z)$ is nonconstant.** Prove that $P(z) = Q(z)$.

Walkthrough —

(a) Assume that $\deg P \geq \deg Q$.

(b) Denote these two set of roots by S_1, S_2. Considering multiplicities, show that

$$2\deg P - |S_1| - |S_2| \leq \deg P' = \deg P - 1,$$

which yields

$$|S_1| + |S_2| > \deg P.$$

(c) Note that $P - Q$ vanishes at the elements of $S_1 \cup S_2$, which has size larger than the degree of $P - Q$.

Solution 1. On the contrary, let us assume that $P \neq Q$. Without loss of generality, let us assume that $\deg P \geq \deg Q$. Let S_1 (resp. S_2) denote the common set of zeroes of P, Q (resp. $P+1, Q+1$). For a polynomial $f(x)$, let us denote its multiset of zeroes by $\mathcal{Z}(f)$.

Note that $\mathcal{Z}(P')$ contains $\mathcal{Z}(P) \setminus S_1$, and $\mathcal{Z}(P')$ also contains $\mathcal{Z}(P+1) \setminus S_2$. Since $\mathcal{Z}(P)$ and $\mathcal{Z}(P+1)$ are disjoint, it follows that

$$2\deg P - |S_1| - |S_2| \leq \deg P' < \deg P - 1,$$

where the final step holds since $\deg P \geq \deg Q$, and one of P, Q is nonconstant. This gives that $|S_1| + |S_2| > \deg P$.

Note that S_1, S_2 are disjoint, and

$$P(z) - Q(z) = (P(z) + 1) - (Q(z) + 1)$$

holds. It follows that $P - Q$ vanishes at $S_1 \cup S_2$, and hence $\deg P \geq |S_1| + |S_2|$. This contradicts the inequality $|S_1| + |S_2| > \deg P$. Consequently, we obtain $P = Q$. ■