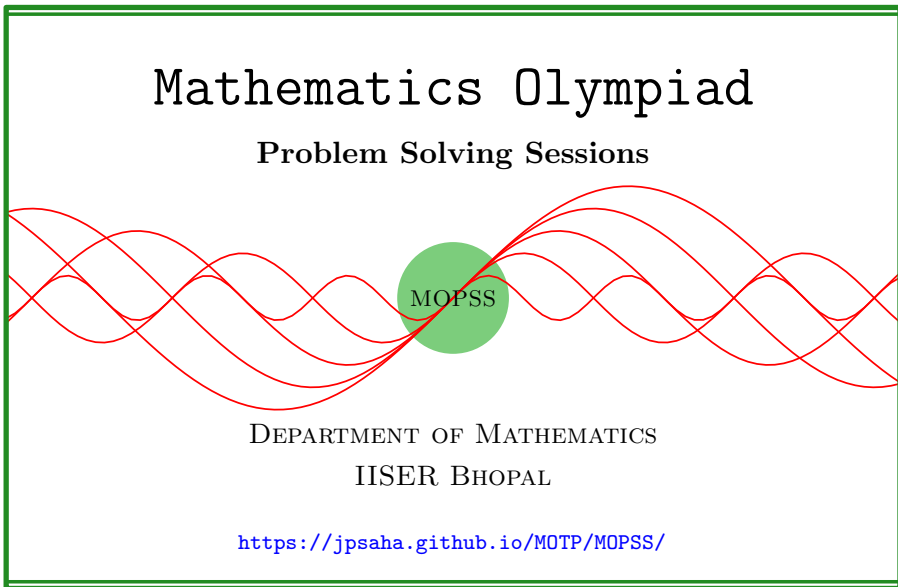


Reduction of the degree by taking a difference

MOPSS

3 June 2024



Suggested readings

- Evan Chen's
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.

- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

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 1.2 Example (Bay Area MO 12 2016 P4) 100

§1 Taking the difference reduces the degree

Note that

$$k^2 - (k - 1)^2 = 2k - 1$$

holds for any integer k . In particular, given a positive integer n , we have

$$\begin{aligned} 2^2 - 1^2 &= 2 \cdot 2 - 1, \\ 3^2 - 2^2 &= 2 \cdot 3 - 1, \\ 4^2 - 3^2 &= 2 \cdot 4 - 1, \\ &\dots = \dots, \\ n^2 - (n - 1)^2 &= 2 \cdot n - 1. \end{aligned}$$

Adding them, it follows that

$$n^2 - 1 = 2(2 + 3 + \dots + n) - (n - 1),$$

which yields

$$\boxed{1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}}.$$

Consider the polynomial $P(x) = x^3$. Note that

$$P(k) - P(k - 1) = 3k^2 - 3k + 1.$$

Using a similar argument as above, it follows that given a positive integer n , we have

$$\begin{aligned} 1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1, \\ 2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1, \\ 3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1, \\ 4^3 - 3^3 &= 3 \cdot 4^2 - 3 \cdot 4 + 1, \\ &\dots = \dots, \\ n^3 - (n - 1)^3 &= 3 \cdot n^2 - 3 \cdot n + 1. \end{aligned}$$

Adding them, it follows that

$$n^3 = 3(1 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n,$$

which yields

$$\begin{aligned} 1 + 2^2 + 3^2 + \cdots + n^2 &= \frac{n^3 - n}{3} + (1 + 2 + 3 + \cdots + n) \\ &= \frac{n^3 - n}{3} + \frac{n(n+1)}{2} \\ &= \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Example 1.1 (India RMO 2013b P3). Consider the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2.$$

Prove that there exists a natural number $n > 2013$ for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

Summary — “Differentiating” a polynomial enough times makes it linear.

Walkthrough — (a) Consider the polynomial $P(k) = k^2$, and the polynomial $Q(k) := P(k) - (k - 1)$.

(b) Since $Q(k)$ is a linear polynomial in k , the difference $R(k) := Q(k) - Q(k - 2)$ is a constant, that is, it does not depend on k .

(c) Note that $R(k)$ is a **± 1 -linear combination**^a of four consecutive squares.

(d) Does this help?

^aWhat does it mean?

Solution 1. Consider the polynomial $P(k) = k^2$, and the polynomial $Q(k) := P(k) - (k - 1)$. Since $Q(k)$ is a linear polynomial in k , the difference $R(k) := Q(k) - Q(k - 2)$ is a constant, that is, it does not depend on k . Indeed, $Q(k) = 2k - 1$, and $R(k) = 4$. Note that $R(k) = k^2 - (k - 1)^2 - (k - 2)^2 + (k - 3)^2$.

Note that 2013^2 and 9999 are congruent modulo 4, that is, they differ by a multiple of 4. Let $m \geq 1$ be an integer such that $9999 = 2013^2 - 4m$. Since

$$-k^2 + (k + 1)^2 + (k + 2)^2 - (k + 3)^2 = -4,$$

it follows that

9999

$$\begin{aligned}
&= 2013^2 \\
&- 2014^2 + 2015^2 + 2016^2 - 2017^2 \\
&- 2018^2 + 2019^2 + 2020^2 - 2021^2 \\
&- \dots \\
&- (2014 + 4(m-1))^2 + (2015 + 4(m-1))^2 + (2016 + 4(m-1))^2 - (2017 + 4(m-1))^2
\end{aligned}$$

It follows that there exists a natural number $n = 2017 + 4(m-1) > 2013$, for which one can change a suitable number of plus signs to minus signs in the expression

$$2013^2 + 2014^2 + 2015^2 + \dots + n^2$$

to make the resulting expression equal to 9999. ■

Example 1.2 (Bay Area MO 12 2016 P4). Find a positive integer N and a_1, a_2, \dots, a_N , where $a_k = 1$ or $a_k = -1$ for each $k = 1, 2, \dots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \dots + a_N \cdot N^3 = 20162016,$$

or show that this is impossible.

Summary — “Differentiating” a polynomial enough times makes it linear.

Walkthrough —

- (a) Consider the polynomial $P(k) := k^3$. Note that $R(k) := P(k) - P(k-1)$ is a quadratic polynomial in k .
- (b) Also note that $S(k) := R(k) - R(k-2)$ is a linear polynomial in k .

Solution 2. Consider the polynomial $P(k) := k^3$. Note that $R(k) := P(k) - P(k-1)$ is equal to $3k^2 - 3k + 1$. Also note that $S(k) := R(k) - R(k-2)$ is equal to $6(2k-2) - 6$. This gives $S(k) - S(k-4) = 48$. It follows that **some ± 1 -linear combination** of any given eight consecutive cubes is equal to 48. More specifically,

$$k^3 - (k-1)^3 - (k-2)^3 + (k-3)^3 - (k-4)^3 + (k-5)^3 + (k-6)^3 - (k-7)^3 = 48,$$

or equivalently,

$$-k^3 + (k+1)^3 + (k+2)^3 - (k+3)^3 - (k+4)^3 + (k+5)^3 + (k+6)^3 - (k+7)^3 = 48.$$

Note that 20162016 is divisible by 3 and 16. Since 3, 16 do not have any common prime factor, it follows that 20162016 is a multiple of 48. Write

$$f(k) = -k^3 + (k+1)^3 + (k+2)^3 - (k+3)^3 - (k+4)^3 + (k+5)^3 + (k+6)^3 - (k+7)^3.$$

Note that

$$f(1) + f(9) + f(17) + \cdots + f(8m - 7) = 20162016,$$

where m denotes the integer $20162016/48$. We conclude that one may take $N = 8m = 20162016/6 = 3360336$ so that the given condition holds. ■