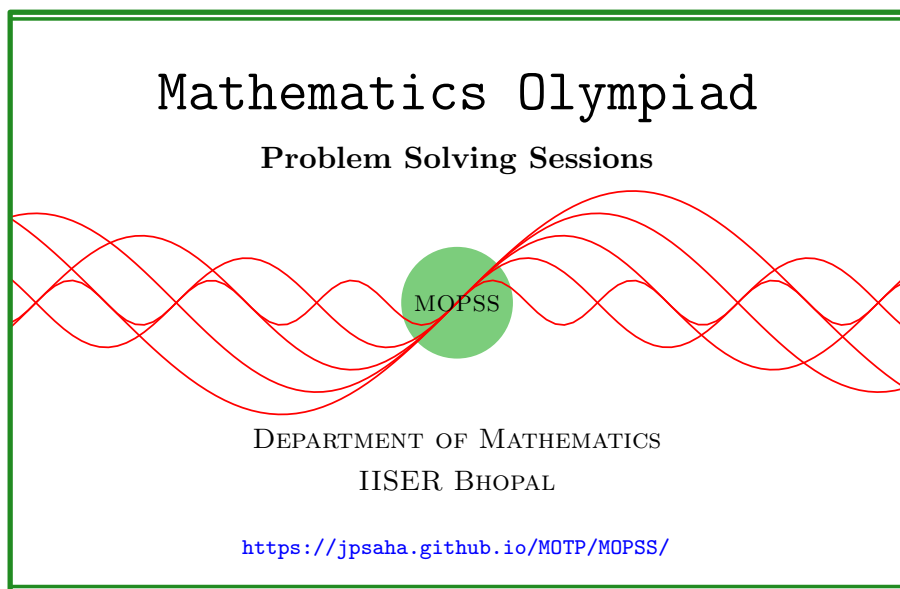


Cubic polynomials

MOPSS

4 June 2024

The logo is enclosed in a double green border. It features the text "Mathematics Olympiad" in a large, black, serif font, with "Problem Solving Sessions" in a smaller, black, serif font below it. A central green circle contains the text "MOPSS" in white. Below the circle, the text "DEPARTMENT OF MATHEMATICS" and "IISER BHOPAL" is written in a black, serif font. At the bottom, a blue URL is provided. The entire logo is decorated with several overlapping red wavy lines that pass behind the central green circle.

Mathematics Olympiad
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<https://jpsaha.github.io/MOTP/MOPSS/>

Suggested readings

- Evan Chen's
 - advice *On reading solutions*, available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
 - *Advice for writing proofs/Remarks on English*, available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

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§1 Cubic polynomials

Example 1.1 (India RMO 1999 P4). If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, show that $p = q = r$.

Solution 1. The given conditions imply

$$p + q + r = 3p, pq + qr + rp = 3q^2, pqr = r^3,$$

which gives

$$q + r = 2p, (q + r)^2 + 2qr = 6q^2, (q + r)qr = 2r^3.$$

Thus

$$(q - r)(5q + r) = r(q + 2r)(q - r) = 0.$$

If $q \neq r$, then we get

$$5q + r = 0, r(q + 2r) = 0,$$

which gives $q = r = 0$. So q, r are equal and hence they are equal to p . ■

Example 1.2 (India RMO 2012b P6). Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution 2. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots¹ of the equation $P(t) = 0$, we obtain

$$x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz = 0,$$

$$y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz = 0,$$

$$z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz = 0.$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x &= ((x + y + z)x^2 - (xy + yz + zx)x + xyz)y \\ &\quad + ((x + y + z)y^2 - (xy + yz + zx)y + xyz)z \end{aligned}$$

¹If it is not clear, then the following equalities may directly be verified.

$$\begin{aligned} &+ ((x + y + z)z^2 - (xy + yz + zx)z + xyz)x \\ = &(x + y + z)(x^2y + y^2z + z^2x) \\ &- (xy + yz + zx)(xy + yz + zx) \\ &+ xyz(x + y + z) \\ = &-(xy + yz + zx)^2 \quad (\text{using } x + y + z = 0) \\ = &-9 \quad (\text{using } xy + yz + zx = -3). \end{aligned}$$

This completes the proof. ■