Cubic polynomials

MOPSS

4 June 2024

Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Cubic polynomials

Example 1.1 (India RMO 1999 P4)**.** If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, show that $p = q = r$.

Solution 1. The given conditions imply

$$p + q + r = 3p, pq + qr + rp = 3q^2, pqr = r^3,$$

which gives

$$q + r = 2p, (q + r)^2 + 2qr = 6q^2, (q + r)qr = 2r^3.$$

Thus

$$(q - r)(5q + r) = r(q + 2r)(q - r) = 0.$$

If $q \neq r$, then we get

$$5q + r = 0, r(q + 2r) = 0,$$

which gives $q = r = 0$. So q, r are equal and hence they are equal to p. ■

Example 1.2 (India RMO 2012b P6)**.** Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution 2. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots[1] of the equation $P(t) = 0$, we obtain

$$\begin{aligned} x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz = 0,\\ y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz = 0,\\ z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz = 0. \end{aligned}$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x = &\big((x + y + z)x^2 - (xy + yz + zx)x + xyz\big)y\\ &+ \big((x + y + z)y^2 - (xy + yz + zx)y + xyz\big)z \end{aligned}$$

[1] If it is not clear, then the following equalities may directly be verified.

$$
\begin{aligned}
&\quad + \big((x+y+z)z^2 - (xy+yz+zx)z + xyz\big)x \\
&= (x+y+z)(x^2y + y^2z + z^2x) \\
&\quad - (xy+yz+zx)(xy+yz+zx) \\
&\quad + xyz(x+y+z) \\
&= -(xy+yz+zx)^2 \qquad (\text{using } x+y+z=0) \\
&= -9 \qquad (\text{using } xy+yz+zx=-3).
\end{aligned}
$$

This completes the proof. ■