Cubic polynomials

MOPSS

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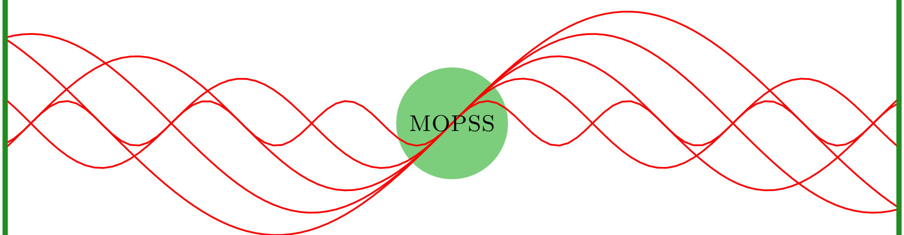
Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Cubic polynomials

Example 1.1 (India RMO 1999 P4)**.** If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, show that $p = q = r$.

Solution 1. The given conditions imply

$$p + q + r = 3p, pq + qr + rp = 3q^2, pqr = r^3,$$

which gives

$$q + r = 2p, (q + r)^2 + 2qr = 6q^2, (q + r)qr = 2r^3.$$

Thus

$$(q - r)(5q + r) = r(q + 2r)(q - r) = 0.$$

If $q \neq r$, then we get

$$5q + r = 0, r(q + 2r) = 0,$$

which gives $q = r = 0$. So q, r are equal and hence they are equal to p. ∎

Example 1.2 (India RMO 2012b P6)**.** Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution 2. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots[1] of the equation $P(t) = 0$, we obtain

$$x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz = 0,$$
$$y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz = 0,$$
$$z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz = 0.$$

[1] If it is not clear, then the following equalities may directly be verified.

Using them, we obtain

$$\begin{aligned}
x^3y + y^3z + z^3x &= \big((x+y+z)x^2 - (xy+yz+zx)x + xyz\big)y \\
&\quad + \big((x+y+z)y^2 - (xy+yz+zx)y + xyz\big)z \\
&\quad + \big((x+y+z)z^2 - (xy+yz+zx)z + xyz\big)x \\
&= (x+y+z)(x^2y + y^2z + z^2x) \\
&\quad - (xy+yz+zx)(xy+yz+zx) \\
&\quad + xyz(x+y+z) \\
&= -(xy+yz+zx)^2 \qquad (\text{using } x+y+z=0) \\
&= -9 \qquad (\text{using } xy+yz+zx=-3).
\end{aligned}$$

This completes the proof. ■

Example 1.3 (India Pre-RMO 2012 P17)**.** Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

See also **??**, USAMO 2014 P1.

Solution 3. Let $P(x)$ denote the polynomial $x^3 + 3x + 5$. Note that

$$\begin{aligned}
&\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right) \\
&= \frac{1}{x_1x_2x_3}(x_1^2+1)(x_2^2+1)(x_3^2+1) \\
&= \frac{1}{x_1x_2x_3}(x_1+i)(x_2+i)(x_3+i)(x_1-i)(x_2-i)(x_3-i) \\
&= \frac{1}{x_1x_2x_3}P(-i)P(i) \\
&= \frac{1}{-5}\left|P(i)\right|^2 \\
&= \frac{1}{-5}\left|5-2i\right|^2 \\
&= -\frac{29}{5}.
\end{aligned}$$

■

Remark. The above argument is elegant and quite useful. One could have also argued that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)$$

$$
\begin{aligned}
&= \left(x_1x_2 + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1x_2}\right)\left(x_3 + \frac{1}{x_3}\right) \\
&= x_1x_2x_3 + \frac{x_1x_3}{x_2} + \frac{x_2x_3}{x_1} + \frac{x_3}{x_1x_2} + \frac{x_1x_2}{x_3} + \frac{x_1}{x_2x_3} + \frac{x_2}{x_3x_1} + \frac{1}{x_1x_2x_3} \\
&= x_1x_2x_3 + + \frac{1}{x_1x_2x_3} + \frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2} + \frac{x_1}{x_2x_3} + \frac{x_2}{x_3x_1} + \frac{x_3}{x_1x_2} \\
&= -5 - \frac{1}{5} + \frac{1}{x_1x_2x_3}\left(x_1^2x_2^2 + x_2^2x_3^3 + x_3^2x_1^2\right) + \frac{1}{x_1x_2x_3}\left(x_1^2 + x_2^2 + x_3^2\right) \\
&= -5 - \frac{1}{5} - \frac{1}{5}\left((x_1x_2 + x_2x_3 + x_3x_1)^2 - 2x_1x_2x_3(x_1 + x_2 + x_3)\right) \\
&- \frac{1}{5}\left((x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)\right) \\
&= -5 - \frac{1}{5} - \frac{1}{5}\left(3^2\right) - \frac{1}{5}\left(-2 \cdot 3\right) \\
&= -5 - \frac{1}{5} - \frac{9}{5} + \frac{6}{5} \\
&= -5 - \frac{4}{5} \\
&= -\frac{29}{5}.
\end{aligned}
$$

Note that this way of arguing would get complication if we had a higher degree polynomial to start with.

Example 1.4 (India RMO 2013a P2). Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:

(a) $f(1) = 0$,

(b) the roots of $g(x) = 0$ are the squares of the roots of $f(x) = 0$.

Find the value of $a^{2013} + b^{2013} + c^{2013}$.

Solution 4. Condition (a) implies that 1 is a root of $f(x)$. Let α, β denote the remaining roots of $f(x)$ in $\mathbb{C}$. It follows that

$$\alpha + \beta + 1 = -a, \alpha + \beta + 1 = b, \alpha\beta = -c.$$

Condition (b) shows that the roots of the polynomial $g(x)$ are $1, \alpha^2, \beta^2$. Consequently,

$$\alpha^2 + \beta^2 + 1 = -b, \alpha^2 + \beta^2 + \alpha^2\beta^2 = c, \alpha^2\beta^2 = -a.$$

hold. Note that

$$
\begin{aligned}
a &= -(\alpha + \beta + 1) \\
&= -b, \\
-b &= (\alpha + \beta + 1)^2 - 2(\alpha + \beta + 1)
\end{aligned}
$$

$$
\begin{aligned}
&= a^2 - 2b \\
&= b^2 - 2b,
\end{aligned}
$$

which shows that $b^2 = b$, implying $b = 0$ or $b = 1$. Also note that $-a = c^2$. If $b = 0$, then $a = 0$ and hence, $c = 0$. By the given condition, we have $c \neq 0$. It follows that $b = 1$ and hence $a = -1$. Using $f(1) = 0$ combined with $a = -1, b = 1$, we obtain $c = -1$. This gives

$$a^{2013} + b^{2013} + c^{2013} = -1.$$

■

Example 1.5 (India RMO 2013e P6)**.** Let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$, where a and b are nonzero real numbers. Suppose that the roots of the equation $P(x) = 0$ are the reciprocals of the roots of the equation $Q(x) = 0$. Prove that a and b are integers. Find the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$.

Solution 5. The reciprocals of the roots of the polynomial $Q(x)$ are the roots of polynomial

$$R(x) := x^3 P\left(\frac{1}{x}\right) = ax^3 + bx^2 + 1.$$

The given condition implies that $R(x) = aP(x)$, that is, $ab = 1, a^2 = b$. This gives $a^3 = 1$. Since a is real, it follows that $a = 1$ and hence $b = 1$. This proves that a, b are integers.

Note that for any integer n,

$$
\begin{aligned}
\gcd(P(n), Q(n)) &= \gcd(n^3 + n^2 + 1, n^3 + n + 1) \\
&= \gcd(n^2 - n, n^3 + n + 1) \\
&= \gcd(n^2 - n, n^3 + n + 1 - (n^2 - n)(n + 1)) \\
&= \gcd(n^2 - n, 2n + 1) \\
&= \gcd(n(n - 1), 2n + 1) \\
&= \gcd(n - 1, 2n + 1) \\
&= \gcd(n - 1, 2(n - 1) + 3) \\
&= \gcd(n - 1, 3).
\end{aligned}
$$

For $n = 2013! + 1$, note that $\gcd(n - 1, 3) = 1$, and hence the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$ is equal to 3. ■

Example 1.6 (India RMO 2016f P3)**.** Find all integers k such that all roots of the following polynomial are also integers:

$$f(x) = x^3 - (k - 3)x^2 - 11x + (4k - 8).$$

Summary — Try to somehow **eliminate** k (for instance, by evaluating f at a suitable integer), and come up with a factorization of an (**nonzero**) integer in terms of the roots of f. This would imply that there are only a finitely many possibilites for the roots of f, provided k is an integer with the stated properties. Under some **non-degeneracy condition**[a], we would obtain some nontrivial algebraic condition on k. Next, the precise choices for k satisfying this condition can be plugged into f to find out for which one of them, the polynomial f possesses integer roots only.

[a] It just means that "**if life is nice!**"

Solution 6. Let k be an integer such that all roots of the polynomial

$$x^3 - (k-3)x^2 - 11x + (4k-8)$$

are integers. Denote the roots of this polynomial by α, β, γ. It follows that

$$\alpha + \beta + \gamma = k - 3, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -11, \quad \alpha\beta\gamma = -(4k-8).$$

Since $\alpha\beta + \beta\gamma + \gamma\alpha = -11$, it follows that at most one of α, β, γ is even. Since the product $\alpha\beta\gamma$ is divisible by 4, it follows that exactly one of α, β, γ is even, and it is a multiple of 4. Reordering α, β, γ if necessary, we assume that 4 divides α. Using $\alpha\beta + \beta\gamma + \gamma\alpha = -11$, it follows that $\beta\gamma \equiv 1 \pmod 4$.

Note that

$$f(2) = 8 - 4k + 12 - 22 + 4k - 8 = -10.$$

This gives $(2-\alpha)(2-\beta)(2-\gamma) = -10$. Since

$$\alpha \equiv 0 \pmod 4, \quad \beta \equiv 1 \pmod 4,$$

holds, it follows that

$$2 - \alpha \equiv 2 \pmod 4, \quad (2-\beta)(2-\gamma) \equiv 1 \pmod 4.$$

This combined with $(2-\alpha)(2-\beta)(2-\gamma) = -10$ shows that $2 - \alpha$ is equal to one of

$$-10, -2, 2, 10,$$

and it is not equal to any of $2, 10$. This gives that $\alpha = 12$ or $\alpha = 4$. Since α is a root of $f(x)$ and $\alpha^2 \neq 4$, we obtain

$$k = \frac{\alpha^3 + 3\alpha^2 - 11\alpha - 8}{\alpha^2 - 4} = \alpha + 3 + \frac{4 - 7\alpha}{\alpha^2 - 4}.$$

Note that $\alpha \neq 12$, otherwise, k would not be an integer. For $\alpha = 4$, we obtain $k = 7 - 2 = 5$.

Finally, note that for $k = 5$,

$$f(x) = x^3 - 2x^2 - 11x + 12,$$

which factorizes into $(x+3)(x-1)(x-4)$, and hence has integer roots.

We conclude that $k = 5$ is the only integer such that the required condition holds. ■

Here is another approach to Example 1.6, which seems to be more general (or, perhaps, *somewhat* more general in sense that it gives an **algorithmic approach** which would work in several other cases. This does not require finding constraints stemming from congruence conditions. Note that coming up with constraints originating from congruence conditions requires to first have an **appropriate modulus** which would provide any presumably useful constraints, without knowing a priori whether any such **modulus** exists.).

Summary — Express k in terms of an integer root of such an equation. Show that such an integer root cannot be **too large**. Use it to narrow down the number of possibilities for k (recall that in absolute values, a higher degree polynomial dominates a lower degree polynomial at arguments which are large enough in absolute values). Check for which of these values the given polynomial has integer roots only.

Solution 7. Let k be an integer such that all the roots of the polynomial

$$f(x) = x^3 - (k-3)x^2 - 11x + (4k-8)$$

are integers[2]. Let α denote an integer root of $f(x)$. Note that $\alpha^2 \neq 4$, and hence,

$$k = \frac{\alpha^3 + 3\alpha^2 - 11\alpha - 8}{\alpha^2 - 4} = \alpha + 3 + \frac{4 - 7\alpha}{\alpha^2 - 4}.$$

This implies that

$$\left|\alpha^2 - 4\right| \leq |4 - 7\alpha|,$$

or equivalently,

$$\begin{cases} -(7\alpha - 4) \leq \alpha^2 - 4 \leq 7\alpha - 4 & \text{if } \alpha \geq \frac{4}{7}, \\ 7\alpha - 4 \leq \alpha^2 - 4 \leq 4 - 7\alpha & \text{if } \alpha \leq \frac{4}{7}. \end{cases}$$

This shows that α lies in

$$([1, 7] \cup [-8, 0]) \setminus \{-2, 2\}.$$

Since k is an integer, it follows that $\alpha^2 - 4$ divides $7\alpha - 4$, and consequently, α is equal to one of

$$-3, 1, 4, 7.$$

Substituting these values of α in

$$k = \alpha + 3 + \frac{4 - 7\alpha}{\alpha^2 - 4},$$

[2]For several of the steps of the argument, it suffices to assume **only** that $f(x)$ has at least one integer root.

it follows that $k = 5$ or $k = 9$. Note that for $k = 9$, we have

$$f(x) = x^3 - 6x^2 - 11x + 28 = (x - 7)(x^2 + x - 4),$$

and not all its roots are integers. It follows that $k = 5$.

Finally, note that for $k = 5$,

$$f(x) = x^3 - 2x^2 - 11x + 12,$$

which factorizes into $(x + 3)(x - 1)(x - 4)$, and hence has integer roots.

We conclude that $k = 5$ is the only integer such that the required condition holds. ■