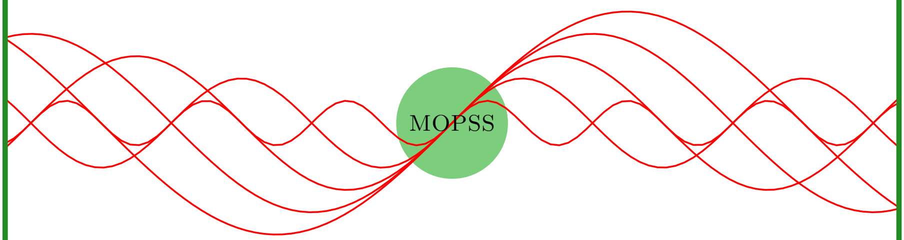
Arithmetic progressions

MOPSS

5 July 2024

Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Arithmetic progressions

Example 1.1. Show that there is a coloring of the positive integers using two colors such that there is no monochromatic infinite arithmetic progression.

We give a solution from this page.

Solution 1. Colour the first one red, the next two blue, the next three red, the next four blue and so on.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 ...

Given an infinite arithmetic progression of positive integers with common difference d, note that any set of consecutive d positive integers contains a term of that progression. Considering a set of at least d consecutive red integers, and a set of at least d consecutive blue integers, it follows that there is no monochromatic infinite arithmetic progression. ■

Example 1.2. [PK74, Problem 47.3] Determine m so that the equation

$$x^4 - (3m+2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression.

Solution 2. Let m be such that the given equation has four real roots in arithmetic progression. Denote the roots of this equation by

$$a - 3d, a - d, a + d, a + 3d.$$

Note that the sum of the roots is equal to 0, which gives $a = 0$. Also note that

$$-3d(-d + d + 3d) - d^2 - 3d^2 + 3d^2 = -(3m+2),$$

which yields $10d^2 = 3m + 2$. Moreover, we also have

$$9d^4 = m^2,$$

which yields $3d^2 = \pm m$. Combining this with $10d^2 = 3m + 2$, we obtain

$$m = \begin{cases} 6 & \text{if } d^2 = \frac{m}{3}, \\ -\frac{6}{19} & \text{if } d^2 = -\frac{m}{3}. \end{cases}$$

Let us determine whether for $m = 6$ or $m = -\frac{6}{19}$, the given equation has four real roots in arithmetic progression. Note that if $m = 6$, then

$$\begin{aligned} x^4 - (3m+2)x^2 + m^2 &= x^4 - 20x^2 + 36 \\ &= (x^2 - 2)(x^2 - 18) \\ &= (x + 3\sqrt{2})(x + \sqrt{2})(x - \sqrt{2})(x - 3\sqrt{2}). \end{aligned}$$

Moreover, if $m = -\frac{6}{19}$, then

$$\begin{aligned} x^4 - (3m+2)x^2 + m^2 &= x^4 - \frac{20}{19}x^2 + \frac{6^2}{19^2} \\ &= \left(x^2 - \frac{2}{19}\right)\left(x^2 - \frac{18}{19}\right) \\ &= \left(x + \frac{3\sqrt{2}}{\sqrt{19}}\right)\left(x + \frac{\sqrt{2}}{\sqrt{19}}\right)\left(x - \frac{\sqrt{2}}{\sqrt{19}}\right)\left(x - \frac{3\sqrt{2}}{\sqrt{19}}\right). \end{aligned}$$

This proves that the required values for m are

$$6, -\frac{6}{19}.$$

■

Example 1.3. [Kos14, Example 2.11] Let x, y be positive integers satisfying the Pell's equation $x^2 - 2y^2 = -1$. Prove that

$$1^3 + 3^3 + 5^3 + \cdots + (2y-1)^3 = x^2y^2.$$

Solution 3. Note that

$$\begin{aligned} &1^3 + 3^3 + 5^3 + \cdots + (2y-1)^3 \\ &= 1^3 + 2^3 + 3^3 + \cdots + (2y)^3 - \left(2^3 + 4^3 + 6^3 + \cdots + (2y)^3\right) \\ &= (y(2y+1))^2 - 8\left(1^3 + 2^3 + 3^3 + \cdots + y^3\right) \\ &= (y(2y+1))^2 - 2\left(y(y+1)\right)^2 \\ &= y^2(4y^2 + 4y + 1 - 2y^2 - 4y - 2) \\ &= y^2(2y^2 - 1) \\ &= x^2y^2. \end{aligned}$$

■

Example 1.4 (India RMO 1994 P1)**.** A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf?

Solution 4. Suppose $1, 2, \ldots, n$ denote the page numbers of the novel and $x, x + 1$ denote the page numbers of the torn leaf. The given condition implies that

$$\frac{1}{2}n(n + 1) = 15000 + 2x + 1. \tag{1}$$

Since $1 \leq x \leq n - 1$, we get

$$15000 + 3 \leq \frac{1}{2}n(n + 1) \leq 15000 + 2n - 1,$$

which shows that

$$n^2 + n - (30000 + 6) \geq 0, \quad n^2 - 3n - (30000 - 2) \leq 0.$$

Thus

$$\frac{-1 + \sqrt{120000 + 25}}{2} \leq n \leq \frac{3 + \sqrt{120000 + 1}}{2}.$$

Now it can be checked that [1]

$$\frac{3 + \sqrt{120000 + 1}}{2} < 175, \quad \frac{-1 + \sqrt{120000 + 25}}{2} > 172.$$

Consequently, n is equal to 173 or 174. Since the number of pages of a novel is an even number, we conclude that $n = 174$. This shows that

$$\begin{aligned} 2x &= 87 \times 175 - 15001 \\ &= 80 \times 175 + 50 \times 25 - 15026, \end{aligned}$$

and hence

$$\begin{aligned} x &= 40 \times 175 + 25 \times 25 - 7513 \\ &= 7000 + 625 - 7513 \\ &= 112. \end{aligned}$$

Consequently, the page numbers on the torn leaf are $112, 113$[2]. ■

Example 1.5 (India RMO 2009 P6)**.** In a book with page numbers from 1 to 100 some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

[1] Since n is close to $\frac{1}{2}\sqrt{120000} = 100\sqrt{3}$, which is close 173. So we tried to bound n using integers close to 173, whose squares can be easily computed since squaring 175 is easy, at least once it is multiplied by 2.

[2] Do you find something wrong with it? One may note that substituting $n = 173$ in Equation (1) would yield $x = 25$, implying that the page numbers on the torn leaf are $25, 26$.

Solution 5. Suppose r pages are torn off. Denote the page numbers of the torn pages by $2n_1 - 1, 2n_1, 2n_2 - 1, 2n_2, \dots, 2n_r - 1, 2n_r$. So

$$\begin{aligned}&1 + 2 + \cdots + 50 \\ &= 4949 + 2n_1 - 1 + 2n_1 + 2n_2 - 1 + 2n_2 + \cdots + 2n_r - 1 + 2n_r,\end{aligned}$$

which gives

$$4(n_1 + \cdots + n_r) - r = 5050 - 4949 = 101. \tag{2}$$

Consequently, we obtain

$$\begin{aligned}101 &= 4(n_1 + \cdots + n_r) - r \\ &\geq 4(1 + 2 + \cdots + r) - r \\ &= 2r(r+1) - r \\ &= r(2r+1),\end{aligned}$$

and hence

$$r \leq \frac{-1 + \sqrt{1 + 4 \cdot 2 \cdot 101}}{4} = \frac{-1 + \sqrt{809}}{4} \leq 7.$$

Moreover, Equation (2) implies that $r \equiv 3 \pmod 4$. Consequently, three pages are torn. ■

Example 1.6 (India RMO 2011b P3)**.** Let $a, b, c > 0$. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and if a^2+b^2, b^2+c^2, c^2+a^2 are in geometric progression, show that $a = b = c$.

Solution 6. The given conditions yield

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}, \quad \frac{b^2+c^2}{a^2+b^2} = \frac{c^2+a^2}{b^2+c^2}.$$

The first condition gives

$$\frac{a-b}{ab} = \frac{b-c}{bc} = \frac{a-c}{b(a+c)}. \tag{3}$$

On the contrary, let us assume that $a \neq c$. Using $a, c > 0$, we get $a^2 \neq c^2$, and hence, the second condition gives

$$\frac{b^2+c^2}{a^2+b^2} = \frac{c^2+a^2}{b^2+c^2} = \frac{a^2-b^2}{c^2-a^2} = \frac{a+b}{c+a}\frac{a-b}{c-a} = -\frac{a+b}{c+a}\frac{a}{a+c},$$

which is impossible since a, b, c are positive. This proves that $a = c$. Using Equation (3), it follows that $a = c$. This completes the proof. ■

Example 1.7 (India RMO 2014a P2). Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let

$$\sum_{i=1}^{n} a_{2i-1}^2 = x, \quad \sum_{i=1}^{n} a_{2i}^2 = y, \quad a_n + a_{n+1} = z.$$

Express d in terms of x, y, z, n.

Solution 7. Note that

$$\begin{aligned} y - x &= (a_2^2 - a_1^2) + (a_4^2 - a_3)^2 + \cdots + (a_{2n}^2 - a_{2n-1}^2) \\ &= d(a_1 + a_2 + a_3 + a_4 + \cdots + a_{2n-1} + a_{2n}). \end{aligned}$$

Note that $a_i + a_{2n-i}$ is equal to $a_n + a_{n+1}$ for any $0 < i < 2n$. Indeed,

$$\begin{aligned} a_i + a_{2n-i} &= a_1 + (i-1)d + a_1 + (2n - i - 1)d \\ &= 2a_1 + (2n-2)d \end{aligned}$$

is independent of i. This shows that

$$y - x = dn(a_n + a_{n+1}) = dnz.$$

which is equal to dnz. Since $a_1, a_2, \ldots, a_{2n}$ is an arithmetic progression of positive real numbers, it follows that $z = a_n + a_{n+1}$ is nonzero, and consequetly,

$$d = \frac{y - x}{nz}.$$

■

References

[**Kos14**] Thomas Koshy. *Pell and Pell-Lucas numbers with applications*. Springer, New York, 2014, pp. xxiv+431. ISBN: 978-1-4614-8488-2; 978-1-4614-8489-9. DOI: 10.1007/978-1-4614-8489-9. URL: http://dx.doi.org/10.1007/978-1-4614-8489-9 (cited p. 3)

[**PK74**] G. Pólya and J. Kilpatrick. *The Stanford Mathematics Problem Book: With Hints and Solutions*. Dover books on mathematics. Teachers College Press, 1974. ISBN: 9780486469249. URL: https://books.google.de/books?id=Q8Gn51gS6RoC (cited p. 2)